Totally Unimodular Matrices

Lecture Notes: ISE/OR/MA 766

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Let A be an $m \times n$ integral matrix with full row rank and b an $m \times 1$ integral vector.

$$\textbf{LP:} \quad \min \{c^T x : \ Ax = b, \ x \geq 0\}$$

$$\textbf{IP:} \quad \min \{c^T x : \ Ax = b, \ x \in Z_+^n\}$$

- Fundamental Theory of LP.
- Basic solution: $x = (x_B, x_N) = (B^{-1}b, 0)$

Observation: If the optimal basis B^* has $\det B^* = \pm 1$, the optimal basic solution $x^* = (B^*)^{-1}b$ is integral.

Why? Cramer's rule. $B^{-1} = \frac{B^\alpha}{\det B}$

Question: Under what conditions do all bases satisfy $\det(B) = \pm 1$?

Definition: A square integral matrix B is unimodular(UM) if $\det B = \pm 1$.

Definition: An integral matrix A is totally unimodular(TUM) if every square nonsingular submatrix of A is UM.

Observation: If A is TUM, then $a_{ij} \in \{-1, 0, 1\}$.

Proposition: Let A be a TUM matrix. Multiplying any row or column of A by -1 results in a TUM matrix.

Proposition: Let A be a TUM matrix. Then the following matrices are all TUM:

$$-A, \quad A^T, \quad [A, I], \quad [A, -A].$$

Definition: A polyhedron is integral if every extreme point is integral.

Proposition: Let A be an $m \times n$ integral TUM matrix. the following polyhedrons are all integral for any vectors b and u of integers:

$$\{x \in R^n : \ Ax \leq b\}$$

$$\{x \in R^n : \ Ax \geq b\}$$

$$\{x \in R^n : \ Ax \leq b, \ x \geq 0\}$$

$$\{x \in R^n : \ Ax = b, \ x \geq 0\}$$

$$\{x \in R^n : \ Ax = b, \ 0 \leq x \leq u\}$$

Theorem: If A is an $m \times n$ integral matrix with full row rank, the following are equivalent:

- Every basis B is UM, i.e., $\det B = \pm 1$.
- The extreme points of $\{x \in R^n : \ Ax = b, \ x \geq 0\}$ are integral for all integral vectors b.
- Every basis has an integral inverse.

Corollary: If A is an $m \times n$ integral matrix, the following are equivalent:

- A is TUM.
- The extreme points of $\{x \in R^n : \ Ax \leq b, \ x \geq 0\}$ are integral for all integral vectors b.
- Every nonsingular submatrix of A has an integral inverse.

$\surd$ Hoffman and Kruskal (1956)

$\surd$ Veinott and Dantzig (1968): a short proof.

- A linear programming problem with a totally unimodular coefficient matrix yields an optimal solution in integers for any objective vector and any integer vector on the right-hand side of the constraints.
- There are non-unimodular problems which yield integral optimal solutions for any objective vector but only certain integer constraint vectors. (Chapter 6–8, Eugene Lawler's Book)
- There are non-unimodular problems which yield integral optimal solutions for any integer constraint vector but only certain objective vectors. (Page 165–168, Eugene Lawler's Book)

Question: Given a matrix A, how do we know it is totally unimodular or not?

Matrices that are not TUM:

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Matrices that are TUM:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- There do not seem to be any easily tested necessary and sufficient conditions for total unimodularity.
- There exist some characterization theorems for totally unimodular matrices. (Ghouila-Houri (1962) and Camion (1965))
- There is also an easily tested set of sufficient (but not necessary) conditions for total unimodularity.

Camion's Characterization

Definition: A matrix A is Eulerian if the sum of the elements in each row and each column is even.

Theorem: A $(0, +1, -1)$ matrix A is totally unimodular if and only if the sum of the elements in each Eulerian square submatrix is a multiple of 4.

$\surd$ Camion (1963a,1963b,1965)

Eulerian Matrices that are not TUM:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Ghouila-Houri's Characterization

Theorem: An $m \times n$ integral matrix A is totally unimodular if and only if for each set $R \subseteq \{1, 2, \cdots, m\}$ can be divided into two disjoint sets R_1 and R_2 such that

$$\sum_{i \in R_1} a_{ij} - \sum_{i \in R_2} a_{ij} \in \{-1, 0, 1\}, \quad j = 1, 2, \cdots, n$$

√ Ghouila-Houri (1962), Berge (1973) and Commoner (1973)

√ Tamir (1976): a short proof based on Camion's theorem.

Theorem: A $(0, +1, -1)$ matrix A is totally unimodular if both of the following conditions are satisfied:

- Each column contains at most two nonzero elements.
- The rows of A can be partitioned into two sets A_1 and A_2 such that two nonzero entries in a column are in the same set of rows if they have different signs and in different sets of rows if they have the same sign.

Corollary: A $(0, +1, -1)$ matrix A is totally unimodular if it contains no more than one $+1$ and no more than one -1 in each column.

TUM matrices

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Definition: A $(0,+1)$ matrix A has the consecutive one's property if for any column j, $a_{ij} = a_{i'j} = 1$ with $i < i'$ implies $a_{lj} = 1$ for $i < l < i'$.

Corollary: A matrix with the consecutive one's property is TUM.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Theorem: The node-arc incidence matrix of a directed graph is TUM.

Why? Exactly one 1 and one -1 in each column.

Integral Circulation Theorem: For the minimum cost circulation problem, if all lower bounds and capacities are integers and there exists a finite optimal circulation, then there exists an integral optimal circulation (whether or not arc costs are integers).

Minimum Cost Circulation Problems

$$\min \ \sum_{i,j} a_{ij}x_{ij}$$

s.t.

$$\sum_j x_{ji} - \sum_i x_{ij} = 0, \quad \forall\ i,$$

$$0 \le l_{ij} \le x_{ij} \le c_{ij}, \quad \forall\ i,\ j.$$

Introducing the slack variables:

$$\begin{aligned} -x_{ij} + r_{ij} &= -l_{ij} \\ x_{ij} + s_{ij} &= c_{ij} \end{aligned}$$

Minimum Cost Circulation Problems

$$\begin{aligned} &\min\ a^T x \\ &\text{s.t.} \\ &\quad A(x, r, s) = b, \\ &\quad x,\ r,\ s\ \geq 0. \end{aligned}$$

$$A = \left(\begin{array}{c|c|c} G & 0 & 0 \\ \hline -I_m & I_m & 0 \\ \hline I_m & 0 & I_m \end{array}\right) \quad b = \left(\begin{array}{c} 0 \\ \hline -l \\ \hline c \end{array}\right)$$

where G is the arc-node incidence matrix of the network.

Matching in Bipartite Graphs

Theorem: A graph is bipartite if and only if its node-edge incidence matrix is totally unimodular.

$\surd$ Asratian *et al. Bipartite Graphs and Their Applications*, Cambridge University Press, 1998. (Page 16, Theorem 2.3.1)

König-Egervary Theorem: Let G be a bipartite graph. The maximum number of arcs in a matching is equal to the minimum number of nodes in a covering of arcs by nodes.

Why? By LP duality.

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