Lecture 6: Totally Unimodular Matrices

Let A be an $m \times n$ integral matrix with full row rank and b an $m \times 1$ integral vector.

LP: $\min \{c^T x : \; Ax = b, \; x \geq 0\}$

IP: $\min \{c^T x : \; Ax = b, \; x \in Z_+^n\}$

Motivation

- Under what conditions that LP has an integral optimal solution?

- Hint:

    - Fundamental Theory of LP.
    - Basic solution: $x = (x_B, x_N) = (B^{-1}b, 0)$

Unimodular Matrix

- A **unimodular matrix** *M* is a square integer matrix with determinant +1 or −1.
- Equivalently, it is an integer matrix that is invertible over the integers, i.e., there is an integer matrix *M'* which is its inverse (these are equivalent under Cramer's rule).
- Thus every equation $Mx = b$, where *M* and *b* are both integer, and *M* is unimodular, has an integer solution.

- Answer:

The optimal bases of LP form a unimodular matrix.

Examples of unimodular matrix

- Unimodular matrices form a group under matrix multiplication, hence the following are unimodular:

- Identity matrix, negative identity matrix
- The inverse of a unimodular matrix
- The product of two unimodular matrices

Question

- How do we know the optimal bases before solving the problem?
- Under what conditions do all bases form a unimodular matrix?
- Answer: Total unimodularity

Total Unimodularity (TUM)

- Definition: A matrix A is totally unimodular if every square non-singular submatrix is unimodular, i.e., every sub-determinant of A is either +1, -1, or 0.

Examples:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Observation: If A is TUM, then $a_{ij} \in \{-1, 0, 1\}$.

Properties

Proposition: Let A be a TUM matrix. Multiplying any row or column of A by -1 results in a TUM matrix.

Proposition: Let A be a TUM matrix. Then the following matrices are all TUM:

$$-A, \quad A^T, \quad [A, I], \quad [A, -A].$$

- ◇ **Theorem 12.7** A matrix A is totally unimodular if and only if any one of the matrices A^T, $-A$, (A, A), (A, I) is totally unimodular.

Main Theorem

- Theorem: The standard form LP with integral right-hand-side vector *b* has an integral optimal solution if its constraint matrix A is totally unimodular.
- Proof: Cramer's Rule

Cramer’s Rule

- Swiss mathematician Gabriel Cramer (1704 - 1752)
- 1750 - Introduction to the Analysis of Lines of Algebraic Curves
- 1729 suggested to Colin Maclaurin (1698 - 1746)
- Theorem:

Let A be an $n \times n$ matrix with $\det(A) \neq 0$. Then the unique solution of the system $Ax = b$ is given by

$$x_i = \frac{\det(A_i)}{\det(A)}, i = 1, 2, ..., n$$

where A_i is obtained by replacing its ith column of A with b.

Expansion of determinant

It is also possible to expand a determinant along a row or column using *Laplace's formula*, which is efficient for relatively small matrices. To do this along row i, say, we write

$$\det(A) = \sum_{j=1}^{n} A_{i,j} C_{i,j} = \sum_{j=1}^{n} A_{i,j} (-1)^{i+j} M_{i,j}$$

where the $C_{i,j}$ represents the i,j element of the matrix cofactors, i.e. $C_{i,j}$ is $(-1)^{i+j}$ times the minor $M_{i,j}$, which is the determinant of the matrix that results from A by removing the i-th row and the j-th column, and n is the length of the matrix.

Extensions

Definition: A polyhedron is integral if every extreme point is integral.

Proposition: Let A be an $m \times n$ integral TUM matrix. the following polyhedrons are all integral for any vectors b and u of integers:

$$\{x \in R^n : \; Ax \leq b\}$$

$$\{x \in R^n : \; Ax \geq b\}$$

$$\{x \in R^n : \; Ax \leq b, \; x \geq 0\}$$

$$\{x \in R^n : \; Ax = b, \; x \geq 0\}$$

$$\{x \in R^n : \; Ax = b, \; 0 \leq x \leq u\}$$

Main Result

Theorem: If A is an $m \times n$ integral matrix with full row rank, the following are equivalent:

- Every basis B is UM, i.e., $\det B = \pm 1$.
- The extreme points of $\{x \in R^n : \ Ax = b, \ x \geq 0\}$ are integral for all integral vectors b.
- Every basis has an integral inverse.

Corollary

Corollary: If A is an $m \times n$ integral matrix, the following are equivalent:

- A is TUM.
- The extreme points of $\{x \in R^n : \; Ax \leq b, \; x \geq 0\}$ are integral for all integral vectors b.
- Every nonsingular submatrix of A has an integral inverse.

$\surd$ Hoffman and Kruskal (1956)

$\surd$ Veinott and Dantzig (1968): a short proof.

More facts

- A linear programming problem with a totally unimodular coefficient matrix yields an optimal solution in integers for any objective vector and any integer vector on the right-hand side of the constraints.
- There are non-unimodular problems which yield integral optimal solutions for any objective vector but only certain integer constraint vectors. (Chapter 6–8, Eugene Lawler's Book)
- There are non-unimodular problems which yield integral optimal solutions for any integer constraint vector but only certain objective vectors. (Page 165–168, Eugene Lawler's Book)

Question

- Given a matrix A, how do we know it is totally unimodular or not?

Matrices that are not TUM:

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Matrices that are TUM:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

What do we know?

- There do not seem to be any easily tested necessary and sufficient conditions for total unimodularity.
- There exist some characterization theorems for totally unimodular matrices. (Ghouila-Houri (1962) and Camion (1965))
- There is also an easily tested set of sufficient (but not necessary) conditions for total unimodularity.

Camion's Characterization

Definition: A matrix A is Eulerian if the sum of the elements in each row and each column is even.

Theorem: A $(0, +1, -1)$ matrix A is totally unimodular if and only if the sum of the elements in each Eulerian square submatrix is a multiple of 4.

$\surd$ Camion (1963a,1963b,1965)

Eulerian Matrices that are not TUM:

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Ghouila-Houri's Characterization

Theorem: An $m \times n$ integral matrix A is totally unimodular if and only if for each set $R \subseteq \{1, 2, \cdots, m\}$ can be divided into two disjoint sets R_1 and R_2 such that

$$\sum_{i \in R_1} a_{ij} - \sum_{i \in R_2} a_{ij} \in \{-1, 0, 1\}, \quad j = 1, 2, \cdots, n$$

√ Ghouila-Houri (1962), Berge (1973) and Commoner (1973)

√ Tamir (1976): a short proof based on Camion's theorem.

Hoffman's Sufficient Conditions

Theorem: A $(0, +1, -1)$ matrix A is totally unimodular if both of the following conditions are satisfied:

- Each column contains at most two nonzero elements.
- The rows of A can be partitioned into two sets A_1 and A_2 such that two nonzero entries in a column are in the same set of rows if they have different signs and in different sets of rows if they have the same sign.

Corollary: A $(0, +1, -1)$ matrix A is totally unimodular if it contains no more than one $+1$ and no more than one -1 in each column.

Examples

TUM matrices

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & -1 \\ -1 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Corollary

Definition: A $(0,+1)$ matrix A has the consecutive one's property if for any column j, $a_{ij} = a_{i'j} = 1$ with $i < i'$ implies $a_{lj} = 1$ for $i < l < i'$.

Corollary: A matrix with the consecutive one's property is TUM.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Integral Circulation Theorem

Theorem: The node-arc incidence matrix of a directed graph is TUM.

Why? Exactly one 1 and one -1 in each column.

Integral Circulation Theorem: For the minimum cost circulation problem, if all lower bounds and capacities are integers and there exists a finite optimal circulation, then there exists an integral optimal circulation (whether or not arc costs are integers).

Minimum Cost Circulation Problem

$$\min \sum_{i,j} a_{ij} x_{ij}$$

s.t.

$$\sum_j x_{ji} - \sum_i x_{ij} = 0, \quad \forall\ i,$$

$$0 \le l_{ij} \le x_{ij} \le c_{ij}, \quad \forall\ i,\ j.$$

Introducing the slack variables:

$$\begin{aligned} -x_{ij} + r_{ij} &= -l_{ij} \\ x_{ij} + s_{ij} &= c_{ij} \end{aligned}$$

Minimum Cost Circulation Problem

$$\min \; a^T x$$

s.t.

$$A(x, r, s) = b,$$

$$x,\; r,\; s \;\geq 0.$$

$$A = \left(\begin{array}{c|c|c} G & 0 & 0 \\ \hline -I_m & I_m & 0 \\ \hline I_m & 0 & I_m \end{array}\right) \quad b = \left(\begin{array}{c} 0 \\ \hline -l \\ \hline c \end{array}\right)$$

where G is the arc-node incidence matrix of the network.

Question

- Why is the matrix A totally unimodular?

$$A = \left(\begin{array}{c|c|c} G & 0 & 0 \\ \hline -I_m & I_m & 0 \\ \hline I_m & 0 & I_m \end{array}\right)$$

Matching and Bipartite Graph

Theorem: A graph is bipartite if and only if its node-edge incidence matrix is totally unimodular.

$\surd$ Asratian *et al. Bipartite Graphs and Their Applications*, Cambridge University Press, 1998. (Page 16, Theorem 2.3.1)

König-Egervary Theorem: Let G be a bipartite graph. The maximum number of arcs in a matching is equal to the minimum number of nodes in a covering of arcs by nodes.

Why? By LP duality.

References

Berge, C., *Graphs and hypergraphs*, North-Holland, Amsterdam, 1973.

Camion, P., Matrices totalement unimodulaires et problèmes combinatoires, *Thèse, Université Libre de Bruxelles*, Février, 1963.

Camion, P., Caractérisation des matrices unimodulaires, *Cahiers Centre Études Rech*, 5, (1963) no.4.

Camion, P., Characterization of titally unimodular matrices, *Proceedings of the American Mathematical Society*, 15(5), (1965) 1068-1073.

Commoner, F.G., A sufficient condition for a matrix to be totally unimodular, *Networks*, 3(4), (1973) 351-365.

Ghouila-Houri, A., Caractérisation des matrices totalement unimodulaires, *Comptes Rendus Hebdomadaires des Séances de l'Académie des Science (Paris)*, 254, (1962) 1192-1194.

References

Hoffman, A.J., Kruskal, J.B., Integral boundary points of convex polyhedral, In Kuhn, H.W., Tucker, A.W., (eds.) *Linear Inequalities and Related Systems*, Princeton University Press, Princeton, 1956, Chapter 13.

Tamir, A., On totally unimodular matrices, *Networks*, 6(4), (1976) 373-382.

Veinott, Jr., A.F., Dantzig, G.B., Integral extreme points, *SIAM Review*, 10(3), (1968) 371-372.