# LECTURE 2: MATHEMATICAL PRELIMINARIES

1. Sets and Relations
2. Graph Theory
3. Linear Programming

# Notations

- Sets and Relations

  - Basic set operations and notations:
    $\in$, $\notin$, $\sim$, $\cup$, $\cap$, $\subseteq$, $\subset$, $\emptyset$, etc

  - $S \subset T$ means $S$ is a *proper subset* of $T$,
    i.e., $S \subseteq T$ but $S \neq T$.

  - $S + e = S \cup \{e\}$
    and
    $S - e = S - \{e\}$

  - *Symmetric Difference*:
    $S \oplus T = (S \cup T) - (S \cap T)$ = the set of all elements contained in $S$ or in $T$, but not both.

  - $|S|$ = the number of elements in $S$ (*cardinality* of $S$.)

  - $\mathcal{P}(S)$ = the *power set* of $S$.

# Sets and Relations

- Suppose that $\mathcal{F}$ is a family of sets.
  $S \in \mathcal{F}$ is *minimal* in $\mathcal{F}$ if there is no $T \in \mathcal{F}$ such that $T \subset S$.
  $S \in \mathcal{F}$ is *maximal* in $\mathcal{F}$ if there is no $T \in \mathcal{F}$ such that $S \subset T$.

- Same concepts of minimality and maximality go for ordered sets.

- If $S$ is a finite set of numbers, min $S$ (max $S$) denotes the numerically smallest (largest) element in $S$.

- Alternative notations for min $A$, where $A = \{a_1, a_2, \ldots, a_n\}$ are:
  $\min\{a_i | 1 \leq i \leq n\}$ or $\min_{1 \leq i \leq n}\{a_i\}$ or $\min_i\{a_i\}$

- $A$ is a matrix whose typical element is $a_{ij}$, written $A = (a_{ij})$. Then
  $\min_j a_{ij}$ = the smallest element in row $i$
  $\max_i a_{ij}$ = the largest element in column $j$.

# Sets and Relations

- Suppose $\leq$ is a total ordering of $A$, i.e., a partial ordering such that for each pair of elements $a$, $b$, in $A$ either $a \leq b$ or $b \leq a$. Then this total ordering induces a *lexicographic ordering* "$\preceq$" of $A^n$, the set of all $n$-tuples of elements of $A$.

  Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$. Then $a \preceq b$ if either $a = b$ or there is some $k$, $1 \leq k \leq n$, such that $a_i = b_i$, $i = 1, 2, \ldots, k-1$, and $a_k < b_k$.

  Suppose $\mathcal{A} = A \cup A^2 \cup A^3 \cup \ldots$. We can define a lexicographic ordering on $\mathcal{A}$ as follows. Let $a = (a_1, a_2, \ldots, a_m)$ and $b = (b_1, b_2, \ldots, b_n)$, where $m \leq n$. Then $a \preceq b$ if $a \preceq (b_1, b_2, \ldots, b_m)$, as defined above, and $b \preceq a$ otherwise.

# Sets and Relations

Or, suppose $\mathcal{A} \subseteq \mathcal{P}(A)$. Let

$$a = \{a_1, a_2, \ldots, a_m\},$$

$$b = \{b_1, b_2, \ldots, b_n\},$$

where $m \leq n$. Assume, without loss of generality, that

$$a_1 \leq a_2 \leq \cdots \leq a_m,$$

and

$$b_1 \leq b_2 \leq \cdots \leq b_n,$$

Then $a \preceq b$ if $(a_1, a_2, \ldots, a_m) \preceq (b_1, b_2, \ldots, b_n)$.

# Graphs and Digraphs

- A *graph* $G = (N, A)$ is a structure consisting of a finite set $N$ of elements called *nodes* and a set $A$ of unordered pairs of nodes called *arcs* . A *directed graph* or *digraph* is defined similarly, except that each arc is an ordered pair, giving it direction from one node to another.
- For both undirected and directed graphs, an arc from node $i$ to node $j$ is denoted by $(i, j)$. An arc $(i, i)$ is called a *loop* . Ordinarily we deal with undirected graphs with no loops and at most one arc between a given pair of nodes $i, j$.

# Graphs and Digraphs

- An arc $(i, j)$ is said to be *incident* to each of the nodes $i$ and $j$, and conversely. Each row of the node-arc incidence matrix is identified with a node and each column with an arc. If the arcs are numbered by the index $k$, then the *incidence matrix* $B = (b_{ik})$ is defined as follows:

$$\begin{aligned} b_{ik} &= 1 \quad \text{if node } i \text{ is incident to arc } k \\ &= 0 \quad \text{otherwise} \end{aligned}$$

- In the case of a directed graph the arc $(i, j)$, directed from $i$ to $j$, is said to be *incident from* $i$ and *incident to* $j$. The arc-node *incidence matrix* $B = (b_{ik})$ is defined as follows:

$$\begin{aligned} b_{ik} &= +1 \quad \text{if arc } k \text{ is incident to node } i \\ &= -1 \quad \text{if arc } k \text{ is incident from node } i \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

# Graphs and Digraphs

- If there exists an arc $(i, j)$ we say that nodes $i$ and $j$ are *adjacent*. For an undirected graph, the *adjacency matrix* $A = (a_{ij})$ is defined as follows:

$$\begin{aligned} a_{ij} &= 1 \quad \text{if there is an arc } (i,j) \text{ between nodes } i \text{ and } j \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

- In the case of a directed graph, if there is an arc $(i, j)$ we say that node $i$ is *adjacent to* node $j$ and node $j$ is *adjacent from* node $i$. The *adjacency matrix* $A = (a_{ij})$ is defined as follows:

$$\begin{aligned} a_{ij} &= 1 \quad \text{if there is an arc } (i,j) \text{ from } i \text{ to } j \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

# Graphs and Diagraphs

- Of special interest is the *bipartite* graph. The nodes of a bipartite graph can be partitioned into two sets $S$ and $T$, such that no two nodes in $S$ or in $T$ are adjacent. If a graph $G = (N, A)$ is bipartite, we commonly denote it as $G = (S, T, A)$ where $N = S \cup T$.
- **Proposition**
  $G$ is a bipartite graph if and only if its nodes can be numbered in such a way that its adjacency matrix takes on the form:

$$A = \left(\begin{array}{c|c} 0 & \overline{A} \\ \hline \overline{A}^T & 0 \end{array}\right)$$

# Graphs and Digraphs

- The *degree* $d_i$ of node $i$ is the number of arcs incident to the node. If $B$ is the incidence matrix,

$$d_i = \sum_k b_{ik}.$$

- In the case of a digraph, the *out-degree* $d_i^{(\mathrm{out})}$ of node $i$ is the number of arcs incident from the node, and the *in-degree* $d_i^{(\mathrm{in})}$ is the number of arcs incident to the node. Note that

$$d_i^{(\mathrm{in})} - d_i^{(\mathrm{out})} = \sum_k b_{ik}$$

# Subgraphs, Cliques and Multigraphs

- The *complete graph* $K_n$ has $n$ nodes any two of which are adjacent. The complete graph has $n(n-1)/2$ arcs. The *complete digraph* on $n$ nodes has $n(n-1)$ arcs. The *complete bipartite graph* $K_{p,q}$ is a bipartite graph $G = (S, T, A)$, with $|S| = p$, $|T| = q$, and $|A| = pq$.
- A graph $G = (N', A')$ is called a *subgraph* of the graph $G = (N, A)$ if $N' \subseteq N$ and $A' \subseteq A$. If $N' \subseteq N$, then the *subgraph* of $G$ *induced* by $N'$ has the node set $N'$ and all arcs $(i, j)$ in $A$ such that both $i$ and $j$ are in $N'$. If a subgraph of $G$ is a complete graph it is a *complete subgraph* . A maximal complete subgraph is called a *clique* .

# Subgraphs, Cliques and Multigraphs

- The *complement* of the graph $G = (N, A)$ is the graph $\overline{G}$ obtained by deleting the arcs of $G$ from the complete graph on the same nodes.
- The *contraction* of an arc $(i, j)$ is accomplished by replacing nodes $i$ and $j$ by a single node $k$. An arc $(k, l)$ is provided in the contracted graph for each arc $(i, l)$ or $(j, l)$ in the original graph, except arc $(i, j)$. The contraction of a graph may well result in a graph with multiple arcs between nodes. Such a structure we call a *multigraph* .

# Connectivity in Graphs

- A *path* between $s$ and $t$, or simply an $(s,t)$ *path*, is a sequence of arcs of the form $(s,i_1),(i_1,i_2),\ldots,(i_k,t)$. If $s,i_1,i_2,\ldots,i_k,t$ are distinct nodes, we say that the path is *minimal* or *without repeated nodes*.
- An $(s,t)$ path is *open* if $s \neq t$ and *closed* if $s = t$. A *cycle* is an $(s,s)$ path containing at least one arc, in which no node except $s$ is repeated. In an ordinary graph, a cycle must contain at least three arcs. A graph which contains no cycles is *acyclic* .
- Two nodes $i$ and $j$ are said to be *connected* if there exists an $(i,j)$ path. A graph $G$ is said to be *connected* if all pairs of nodes are connected. A *component* of a graph $G$ is a maximal connected subgraph. A graph is connected if and only if it has exactly one component.

# Connectivity in Graphs

- **Proposition**
  If a graph $G$ has $p$ components, then its nodes can be numbered in such a way that its adjacency matrix takes on the block diagonal form

$$A = \begin{pmatrix} \boxed{A_1} & 0 & 0 & 0 \\ 0 & \boxed{A_2} & 0 & 0 \\ 0 & 0 & \Box & 0 \\ 0 & 0 & 0 & \boxed{A_p} \end{pmatrix}$$

- A *tree* is a connected acyclic graph.

# Connectivity in Graphs

- **Proposition**

The following statements are equivalent for a graph $G$ with $n$ nodes:

(1) $G$ is a tree.

(2) Every two nodes of $G$ are connected by a unique path.

(3) $G$ is connected and has $n-1$ arcs.

(4) $G$ is acyclic and has $n-1$ arcs.

(5) $G$ is acyclic and if any two nonadjacent nodes of $G$ are joined by an arc $e$, then $G+e$ has exactly one cycle.

(6) $G$ is connected, is not $K_n$ for $n \geq 3$, and if any two nonadjacent nodes of $G$ are joined by a new arc $e$, then $G+e$ has exactly one cycle.

# Connectivity in Graphs

- A *tree* in $G$ is a connected acyclic subgraph on the nodes of $G$. A *forest* in $G$ is an acyclic subgraph on the nodes of $G$. A maximal forest in a connected graph is a *spanning* tree.
- **Theorem**
  Every maximal forest in a graph with $n$ nodes and $p$ components contains $n - p$ arcs.
- **Theorem**
  *(C. W. Borchardt)* $K_n$ contains $n^{n-2}$ distinct spanning trees.

# Connectivity in Digraphs

- A *directed path* from $s$ to $t$, or simpy an $(s,t)$ *path* , is a sequence of arcs from $s$ to $t$, where the $p$th arc is incident to the same node from which the $(p+1)$st arc is incident. A *directed cycle* is a minimal nonempty closed directed path.
- A node $i$ is said to be *connected* to node $j$, and $j$ is said to be *connected from* $i$ if there exists an $(i,j)$ path. A digraph $G$ is said to be *strongly connected* if, for all pairs of nodes $i$ and $j$, $i$ is connected to $j$ and $j$ is connected to $i$. A *strong component* of a graph $G$ is a strongly connected subgraph of $G$ which is maximal.

# Connectivity in Digraphs

◇ **Proposition**

If a directed graph $G$ has $p$ strong components, then its nodes can be numbered in such a way that its adjacency matrix takes on the form

$$A = \begin{pmatrix} \boxed{A_1} & & & \\ 0 & \boxed{A_2} & & \\ 0 & 0 & \boxed{\phantom{A}} & \\ 0 & 0 & 0 & \boxed{A_p} \end{pmatrix}$$

where the entries above the block diagonal submatrices are 0's and 1's.

# Connectivity in Digraphs

- A *directed tree* is either rooted to a node or from a node. A *tree rooted from node $i$* is a tree in which the in-degree of $i$ is zero, and the in-degree of each of the other nodes is at most one. A *tree rooted to node $i$* is a tree in which the out-degree of $i$ is zero and the out-degree of the other nodes is at most one. A *directed spanning tree* is just as its name suggests.
- A directed graph is called *acyclic* if it contains no directed cycles.

# Connectivity in Digraphs

- **Proposition**
  A directed graph is acyclic if and only if its nodes can be numbered in such a way that for all arcs $(i, j)$, $i < j$.

- RENUMBERING THE NODES OF AN ACYCLIC DIGRAPH

  *Step 0 (Start)*

  Set $d_j^{(\mathrm{in})} = \sum_{i=1}^{n} a_{ij}, \quad j = 1, 2, \ldots, n \quad ,$

  Set $N = \{1, 2, \ldots, n\}$.

  Set $m = 1$.

  *Step 1 (Detection of Node with Zero In-Degree)*

  Find $k \in N$ such that $d_k^{(\mathrm{in})} = 0$. If there is no such $k$, stop; the digraph is not acyclic.

  Set $v(k) = m$.

  Set $m = m + 1$.

  Set $N = N - k$.

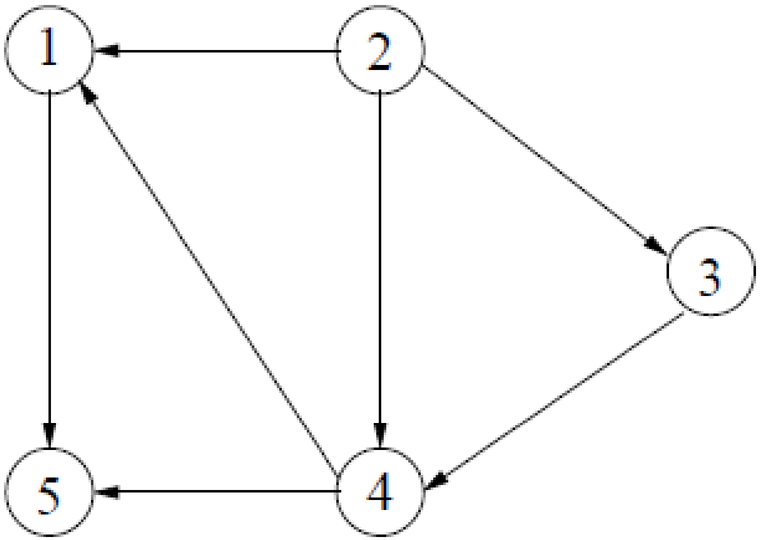
  If $N = \emptyset$, stop; the computation is completed.

  *Step 2 (Revision of In-Degrees)*

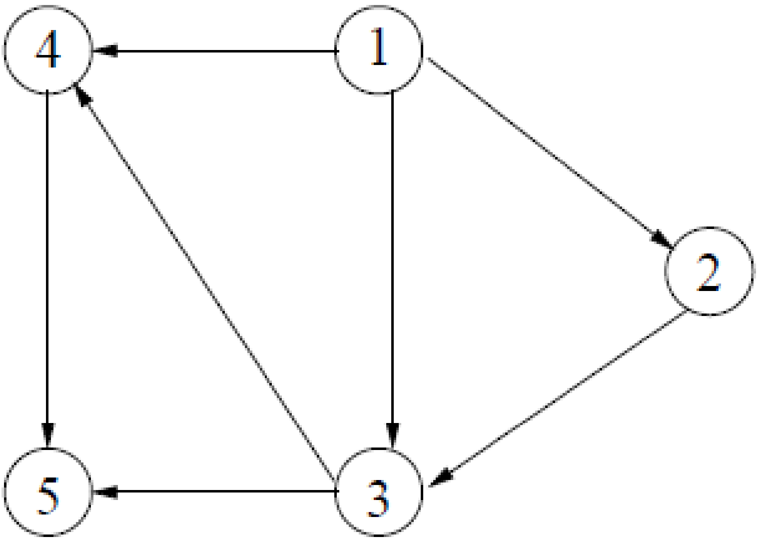
  Set $d_j^{(\mathrm{in})} = d_j^{(\mathrm{in})} - a_{kj}, \quad$ for all $j \in N$.

  Return to Step 1.

# Example




Acyclic ?




Proposition 6.2 ?

# Questions

1. Is Proposition 6.2 valid?
2. How to re-number nodes?
3. Does the re-numbering algorithm work?
   (a) Is each step valid?
   (b) Will it stop?
   (c) When it stops, does it provide a solution?
4. How efficient is the algorithm?
5. How to identify an acyclic diagraph from the adjancency matrix?

# Part of Answers

a). Computation work at each step

Step 0: *n(n-1) additions* plus minors

$\approx O(n^2)$

Step 1: *at most n comparisions to find k* plus minors

$\approx O(n)$

Step 2: *at most (n-1) substractions*

$\approx O(n)$

b). How many iterations ?

Step 0: *once only* $\approx O(n^2)$

Steps 1 & 2: *at most n times* $\approx O(n^2)$

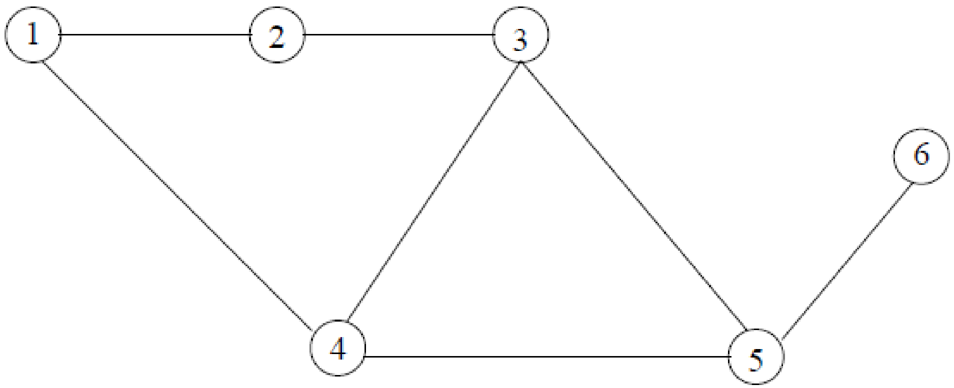
Total complexity $O(n^2)$.

5. Adjancency matrix: upper-triangular.

# Cocycles and Directed Cocycles

- Let $G = (N, A)$ be a graph, or a directed graph in which the directions of the arcs are ignored. A subset $C \subseteq A$, such that $G' = (N, A - C)$ contains more components than $G$, is a *separating set* of $G$. A minimal separating set is a *cocycle* of $G$.
- Given an arbitrary node partition $S$, $T$, the set of arcs extending between $S$ and $T$ is not necessarily a cocycle. We call a separating set determined by such a partition a *cutset* and we may refer to it by any one of the node partitions $S$, $T$ which determines it. An $(s, t) - cutset$ is any cutset $(S, T)$, where $s \in S$ and $t \in T$.

# Example




Separating sets:

$$
\begin{aligned}
C_1 &= \{(1,2),(1,4)\} \\
C_2 &= \{(1,2),(2,3),(1,4)\} \\
C_3 &= \{(2,3),(1,4)\} \\
C_4 &= \{(1,2),(3,4),(4,5),(5,6)\} \\
C_5 &= \{(1,2),(3,4),(4,5)\} \\
C_6 &= \{(5,6)\}
\end{aligned}
$$

# Example

| Cocycles: | Cutsets |
|---|---|
| $C_1$ ? | $S = \{1, 4, 6\}$ |
| $C_3$ ? | $T = \{2, 3, 5\}$ |
| $C_5$ ? | $C_4$ is an $(S, T)$ cutset |
| $C_6$ ? | but $C_4 = C_5 \cup C_6$ |
| $\vdots$ | |

# Cocycles and Directed Cocycles

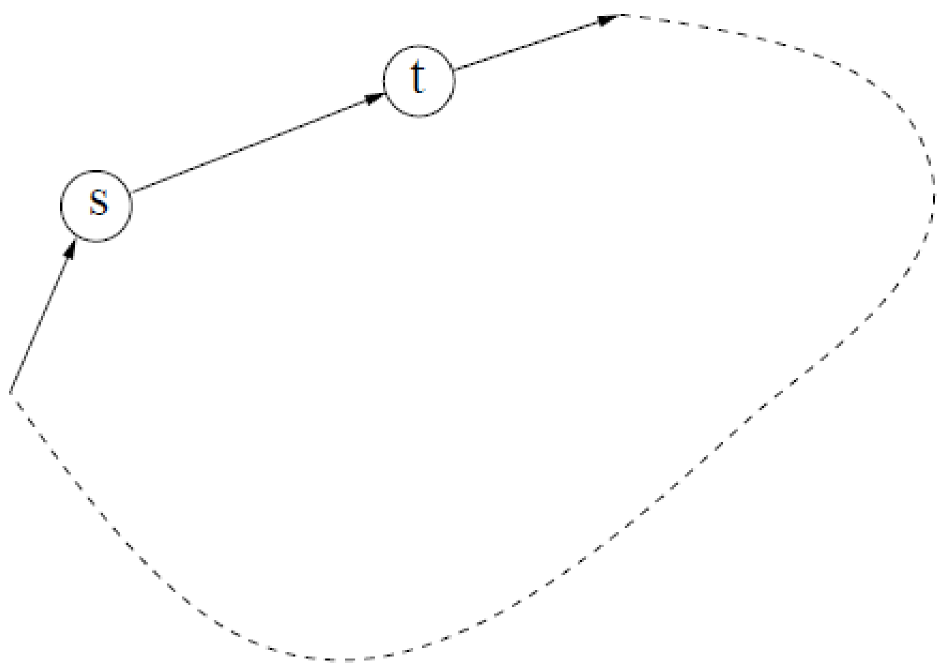
- **Proposition**
  Every cutset is a union of disjoint cocycles.
- **Theorem** *(Minty)*
  Let $G$ be a directed graph with a distinguished arc $(s, t)$. Then, for any painting of the arcs green, yellow, and red, with $(s, t)$ painted yellow, exactly one of the following alternatives holds:

(1) $(s, t)$ is contained in a cycle of yellow and green arcs, in which all yellow arcs have the same direction.

(2) $(s, t)$ is contained in a cocycle of yellow and red arcs, in which all yellow arcs have the same direction.

# Proof of Minty's Theorem

| | |
|---|---|
| Green: | two-way street |
| Yellow: | one-way street |
| Red: | blocked street |



# Proof of Minty's Theorem

Case 1: There is a directed path from $t$ to $s$. So we have a cycle and *condition 1 is met.*

Case 2: No directed path from $t$ to $s$. Let
$T = \{$ all nodes accessible from $t\}$,
$S = N - T$, $(s \in S\ ?)$
So we have a cutset of reds and yellows in the same direction as $(s, t)$. By Proposition 7.1, there is such a cocyle and *condition 2 is met.*

# Eulerian and Hamiltonian Graphs

- The general question, for a given graph $G$, is whether there exists a closed path which contains each arc exactly once. Such a path, if it exists, we call an *Euler path*, and we say the graph is an *Euler graph*, or *Eulerian*.
- **Theorem**
  A graph (or multigraph) $G$ is Eulerian if and only if $G$ is connected and each node of $G$ has even degree.

# Eulerian and Hamiltonian Graphs

- We call a cycle that passes through each node of a graph exactly once a *Hamilton cycle*, and the graph which contains it a *Hamilton graph*, or *Hamiltonian.*
- **Theorem** *(Chvátal)*
  Let $G$ be a graph with $n \geq 3$ nodes and no loops or multiple arcs in which the nodes are numbered so that $d_1 \leq d_2 \leq \cdots \leq d_n$. $G$ is Hamiltonian if

$$d_k \leq k \Rightarrow d_{n-k} \geq n-k, \quad \text{for } 1 \leq k \leq \frac{n}{2}.$$

# Intersection Graphs

- Let $S$ be a set and $\mathscr{S} = \{S_1, S_2, \ldots, S_n\}$ be a family of distinct nonempty subsets of $S$ whose union is $S$. The *intersection graph of* $\mathscr{S}$ is a graph whose nodes are identified with sets in $\mathscr{S}$, with $S_i$ and $S_j$ adjacent whenever $i \neq j$ and $S_i \cap S_j \neq \emptyset$. A graph $G$ is an *intersection graph on* $S$ if there exists a family $\mathscr{S}$ of subsets of $S$, with $G$ the intersection graph of $\mathscr{S}$.

- **Theorem**

  Every graph $G = (N, A)$ is an intersection graph.

# Line Graphs

- For a given graph $G = (N, A)$, we can let $S = N$ and $\mathscr{S} = A$. The intersection graph of $A$ is called the *line graph of* $G$, denoted $L(G)$. A graph $G'$ is called a *line graph* if there exists a graph $G$, with $G' = L(G)$. Sometimes $L(G)$ is called the "arc-to-node dual" of $G$.
- **Theorem**
  $G$ is a line graph if and only if the arcs of $G$ can be partitioned into complete subgraphs in such a way that no node lies in more than two of the subgraphs.

# Eulerian and Hamiltonian Graphs

- **Theorem**
  $G$ is Eulerian if and only if $L(G)$ is Hamiltonian.

- **Theorem**
  If $G$ is Eulerian, then $L(G)$ is Eulerian.

# Linear Programming

$$
\begin{aligned}
&\text{minimize} \\
&\quad z = c^T x \\
&\text{subject to} \\
&\quad Ax = b, \\
&\quad x \geq 0,
\end{aligned}
$$

- Any $m$ linearly independent columns of $A$ will be referred to as a *basis* of the linear system $Ax = b$.

# Linear Programming

- If, for a basis $B$, we suppress the $n - m$ secondary variables, the linear system $Bx^B = b$ is obtained, and this system possesses a unique solution $x^B = B^{-1}b$. The *basic solution associated with* $B$ is defined as $x^B = B^{-1}b, x^N = 0$, but often we refer to the basic solution as simply $x^B$. A basic solution $x^B$ which is feasible (i.e., $x^B \geq 0$) we call a *basic feasible solution* and a basic solution which is optimal we call a *basic optimal solution* .
- **Theorem**

  If there exists a feasible solution to the $LP$, there exists a basic feasible solution.
- **Theorem**

  If there exists an optimal solution to the $LP$, there exists a basic optimal solution.

# Linear Programming

- **Proposition**
  A system of linear inequalities determines a convex polyhedron with integer vertices if and only if, for all possible choices of an objective function, there exists a finite optimal solution in integers.

# Duality Theory

| Primal Problem | Dual Problem |
| --- | --- |
| Minimize $z = \sum_{j=1}^{n} c_j x_j$ | Minimize $w = \sum_{i=1}^{m} (-b_i) u_i$ |
| Subject to | Subject to |
| $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$ | $u_i \geq 0$ |
| $\sum_{j=1}^{n} a_{ij} x_j = b_i$ | $u_i$ unrestricted |
| $x_j \geq 0$ | $\sum_{i=1}^{m} (-a_{ij}) u_i \geq -c_j$ |
| $x_j$ unrestricted | $\sum_{i=1}^{m} (-a_{ij}) u_i = -c_j$ |

# Linear Duals

- The following pairs of problems are duals:

| | |
|---|---|
| minimize $c^T x$ | maximize $u^T b$ |
| subject to | subject to |
| $Ax \geq b$ | $uA \leq c$ |
| $x \geq 0$ | $u \geq 0$ |

| | |
|---|---|
| minimize $c^T x$ | maximize $u^T b$ |
| subject to | subject to |
| $Ax = b$ | $uA \leq c$ |
| $x \geq 0$ | $u$ unrestricted |

| | |
|---|---|
| minimize $c^T x$ | maximize $u^T b$ |
| subject to | subject to |
| $Ax = b$ | $uA = c$ |
| $x$ unrestricted | $u$ unrestricted |

# Duality Theory

- **Theorem** *(Weak Duality)*
  If $\overline{x}$ and $\overline{u}$ are feasible solutions to dual problems, then $c^T\overline{x} \geq \overline{u}^T b$.
- **Corollary**
  If $\overline{x}$ and $\overline{u}$ are feasible solutions to dual problems and $c^T\overline{x} = \overline{u}^T b$ then $\overline{x}$ and $\overline{u}$ are optimal solutions.
- **Theorem** *(Strong Duality)*
  If either problem of a dual pair of problems has a finite optimum, then the other does also and the two optimal objective values are equal; if either has an unbounded optimum, the other has no feasible solution.

# Duality Theory

- **Theorem** *(Orthogonality of Optimal Solutions)*
  If $\overline{x}$ and $\overline{u}$ are feasible solutions to

| | |
|---|---|
| minimize $c^T x$ | maximize $u^T b$ |
| subject to | subject to |
| $Ax \geq b$ | $uA \leq c$ |
| $x \geq 0$ | $u \geq 0$ |

  then $\overline{x}$ and $\overline{u}$ are optimal if and only if $(\overline{u}A - c)^T \overline{x} = \overline{u}^T(A\overline{x} - b) = 0$. That is, if and only if, for $j = 1, 2, \ldots, n$,

$$\overline{x}_j > 0 \quad \text{implies} \quad \sum_{i=1}^{m} \overline{u}_i a_{ij} = c_j$$

  and, for $i = 1, 2, \ldots, m$,

$$\overline{u}_i > 0 \quad \text{implies} \quad \sum_{j=1}^{n} a_{ij} \overline{x}_j = b_i.$$