Issued: 02/27/2023 **OR/MA 706 HW #4** Due: Tuesday Class 03/07/2023

Given the following two-dimensional real-valued functions:

$$\begin{aligned} f(\boldsymbol{x}) &= (x_1^2 + x_2^2 - 1)^2, \\ g_1(\boldsymbol{x}) &= -x_1 - 1, \\ g_2(\boldsymbol{x}) &= x_1 - 1, \\ g_3(\boldsymbol{x}) &= -x_2 - 1, \\ g_4(\boldsymbol{x}) &= x_2 - 1. \end{aligned}$$

Consider the following problems:

$$\begin{array}{ll} \min & f(\boldsymbol{x}) \\ s.t. & g_i(\boldsymbol{x}) \leq 0, \ i = 1, 2, 3, 4, \end{array} \tag{P1}$$

and

$$\begin{array}{ll} \max & f(\boldsymbol{x}) \\ s.t. & g_i(\boldsymbol{x}) \leq 0, \ i = 1, 2, 3, 4, \end{array} \tag{P2}$$

Problem 1: [10 pts] Draw $graf(f)$ in the three-dimensional space E^3.

Problem 2: [5 pts] Draw the feasible domain $\mathscr{F}$ of (P1) and (P2) in the two-dimensional space E^2.

Problem 3: [10 pts] Draw the contour map of $f(\boldsymbol{x}) = C$ in E^2 for $C = 0, 1, 4$.

Problem 4: [20 pts] Given two points $\boldsymbol{x}^1 = (\frac{1}{3}, \frac{1}{2})^{\mathrm{T}} \in \mathscr{F}$, and $\boldsymbol{x}^2 = (-\frac{8}{10}, -\frac{9}{10})^{\mathrm{T}} \in \mathscr{F}$. For $j = 1, 2$, draw the vectors of the gradient and negative gradient at $\boldsymbol{x}^j$, i.e., $\nabla f(\boldsymbol{x}^j)$ and $-\nabla f(\boldsymbol{x}^j)$, on the contour map in Problem 3, respectively. Verify that $f(\boldsymbol{x}^j)$ will locally increase along the direction of $\nabla f(\boldsymbol{x}^j)$, and $f(\boldsymbol{x}^j)$ will locally decrease along the direction of $-\nabla f(\boldsymbol{x}^j)$.

Problem 5: [5 pts] Identify all local minimum and maximum solutions using the results of Problems 1-4.

Problem 6: [25 pts] At each local optimal solution obtained from Problem 5, find the corresponding $\boldsymbol{\lambda} \in \mathbb{R}^4$ with $\lambda_i \geq 0$, $i = 1, 2, 3, 4$ such that $\nabla f(\boldsymbol{x}) + \sum_{i=1}^4 \lambda_i \nabla g_i(\boldsymbol{x}) = 0$.

Problem 7: [25 pts] At each local optimal solution obtained from Problem 5 and its corresponding $\boldsymbol{\lambda}$ obtained from Problem 6, compute its Hessian matrix $L(\boldsymbol{x}) = F(\boldsymbol{x}) + \boldsymbol{\lambda}^{\mathrm{T}} G(\boldsymbol{x}) = \nabla^2 f(\boldsymbol{x}) + \sum_{i=1}^4 \lambda_i \nabla^2 g_i(\boldsymbol{x})$. Show that $L(\boldsymbol{x})$ is positive semi-definite at all local minimum solutions and negative semi-definite at all local maximum solutions.