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LINEAR FRACTIONAL FUNCTIONALS PROGRAMMING

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The present paper deals with the problem of maximizing the ratio of two linear functions subject to a set of linear equalities and nonnegativity constraints on the variables. The problem is attacked directly, beginning with a basic feasible solution and showing the conditions under which the solution can be improved. Conditions for optimality criteria are established. The method followed is similar to 'simplex technique' in linear programming.

THIS paper deals with certain characteristics of programming with linear fractional functionals. Such programming problems have recently been a subject of wide interest in nonlinear programming. An example of fractional programming was dealt with by J. R. ISBELL AND W. H. MARLOW.[1] Five programming games have this form when forces are in the field and the decision means a distribution of fire of each type of unit among the several possible types of targets. The method explained here will be useful in the solution of economic problems in which the different economic activities utilize fixed resources in proportion to the level of their values. The purpose of optimization, however, is not the definition of a revenue, allocation, or economizing extremum (as in linear programming), but the extremum of a specific index number, usually the most favorable ratio of revenues and allocations.

In a recent paper[2] CHARNES AND COOPER solved a programming problem with linear fractional functionals by resolving it into two linear programming problems. For example, suppose it is required to:

$$\begin{aligned} &\text{maximize} && R(X)=(c'X+\alpha)/(d'X+\beta),\\ &\text{subject to} && AX=b,\\ &&& X\geqq 0, \end{aligned} \qquad (1)$$

where (i) X, c, and d are $n\times 1$ vectors,

(ii) A is $m\times n$ matrix, $A=\|aij\|$ $(i=1,\cdots,m; j=1,\cdots,n)$

(iii) b is $m\times 1$ vector,

(iv) c', d' denote the transpose of vectors c and d,

and (v) α, β are scalar constants.

Further it is assumed that the constraints of (1) are regular, so that the solution set

$$S=\{X/AX=b,\ X\geqq 0\}\quad \text{is nonempty and bounded.}$$

For this purpose they solved the following two ordinary linear programming problems (under transformation $Y=tX$).

$$\begin{array}{lll} \text{maximize} & c'Y+\alpha t & \\ \text{subject to} & AY-bt=0, & \\ & d'Y+\beta t=1, & (2) \\ & Y,\ t\geqq 0, & \\ \text{and maximize} & -c'Y-\alpha t, & \\ \text{subject to} & AY-bt=0, & \\ & -d'Y-\beta t=1, & (3) \\ & Y,\ t\geqq 0. & \end{array}$$

The object of this paper is to give an algorithm for the solution of programming with linear fractional functionals without reducing it to linear programming problems. In order to arrive at the Algorithm we have first to prove Theorem 1, given below; thereafter the algorithm for solving programming with linear fractional functionals is investigated. All this is possible if we further assume that set of any 'm' columns of matrix 'A' are linearly independent.

WE SHALL begin by proving the following theorem:

THEOREM 1. *The maximum of $R(X)$ will occur at the basic feasible solution of $AX=b$ and $X\geqq 0$.*

To prove this theorem we shall require the following Lemma established in reference 2:

LEMMA. *Every Y, t satisfying the constraints*

$$AY-bt=0,$$
$$d'Y+\beta t=\gamma,$$
$$Y,\ t\geqq 0\ (\gamma\neq 0 \text{ is a specified number}),$$

has $t>0$.

Proof. The maximum of $c'Y+\alpha t$ or $-c'Y-\alpha t$ will occur at the basic feasible solution of (2) or (3) respectively.

Let us consider the problem of maximizing

$$c'Y+\alpha t,$$

subject to (2).

Any basic feasible solution of (2) consistent with the regularity of S has always $t>0$ strictly. Now (2) involves $(n+1)$ variables and $(m+1)$ constraints. The number of basic variables in any basic feasible solution will be $m+1$ and one of the basic variables will always be t (S being regular). Let, in any basic feasible solution, other basic variables be $y_1\,y_2,\ \cdots,\ y_m$. Without loss of generality we can say that they are the first m variables of (2).

Now the matrix

$$B = \begin{Vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{Vmatrix}$$

formed by the coefficients of the corresponding variables† $x_1, x_2, \cdots, x_m$ [which with $x_k=0, k=m+1, \cdots, n$ provides a feasible solution of (1)] in (1) is nonsingular, since any 'm' columns of A are linearly independent.

Thus we conclude that to every basic feasible solution of (2) consistent with the regularity of S, there corresponds a basic feasible solution of (1) connected by the relation $Y=tX$.

We know in reference 2 thar if (Y^*, t^*) is the optimal solution of $c'Y+\alpha t$ subject to (2), then $Y^*/t^*=X^*$ is the optimal solution of $R(X)$ subject to (1). (Y^*, t^*) is the basic feasible solution of (2); therefore X^* will be the basic feasible solution of (1).

Hence the theorem is proved.

ALGORITHM

Now WE solve the linear fractional functionals programming problem.

Maximize $\quad (c'X+\alpha)/(d'X+\beta)=R(X)$

subject to $\quad AX=b,$

$$X \geqq 0,$$

with the additional assumption *the denominator is positive for all feasible solutions.*

Let X_B be the initial basic feasible solution: such that

$$BX_B=b,$$

or $\quad X_B=B^{-1}b,$

where $\quad B=(b_1, b_2, \cdots, b_m),$

$$X_B \geqq 0.$$

Further let $\quad z^1=c_B'X_B+\alpha,$

and $\quad z^2=d_B'X_B+\beta,$

where c_B' and d_B' are the vectors having their components as the coefficients associated with the basic variables in the numerator and the denominator of the objective function respectively. In addition we assume that for this basic feasible solution

$$u_j=B^{-1}a_j,$$

$$z_j^1=c_B'u_j,$$

$$z_j^2=d_B'u_j,$$

are known for every column a_j of A not in B.

We now wish to examine the possibility of finding another basic feasible solution with improved value of $z=z^1/z^2$; we shall confine our attention to those basic feasible solutions in which only one column of B is changed. As in reference 4 if

† x_γ corresponds to the variable y_γ.

the new basic feasible solution is denoted by $\hat{X}_B$, then $\hat{X}_B=\hat{B}^{-1}b$,

where
$$\hat{B}=(\hat{b}_1, \hat{b}_2, \cdots, \hat{b}_m),$$

i.e., a new nonsingular matrix obtained from B by removing b_γ and replacing it by a_j. The columns of the new matrix $\hat{B}$ are given by

$$\hat{b}_i=b_i, \qquad (i\neq\gamma)$$
$$\hat{b}_\gamma=a_j.$$

We obtain the value of the new basic variables in terms of the original ones and the u_{ij}, i.e.,

$$\left.\begin{aligned}\hat{x}_{Bi}&=x_{Bi}-x_{B\gamma}(u_{ij}/u_{\gamma j}),\\ \hat{x}_{B\gamma}&=x_{B\gamma}/u_{\gamma j}=\theta(\text{say}),\end{aligned}\right\} \qquad (i\neq\gamma)$$

where
$$a_j=\textstyle\sum_{i=1}^{i=m} u_{ij}\, b_i.$$

Our main interest was in finding a new basic feasible solution with an improved value of the objective function. Having found a new basic feasible solution we must determine whether 'z' is improved. The value of the objective function for the original basic feasible soultion is $z=z^1/z^2$. Let the new value of the objective function be

$$\bar{z}=\bar{z}^1/\bar{z}^2.$$

As in reference 4 we have

$$\bar{z}^1=z^1+\theta(c_j-z_j^1),$$

and
$$\bar{z}^2=z^2+\theta(d_j-z_j^2);$$

here z_j^1 and z_j^2 refer to the original basic feasible solution.

The value of the objective function will improve if

$$\bar{z}>z,$$

or
$$[z^1+\theta(c_j-z_j^1)]/[z^2+\theta(d_j-z_j^2)]>z^1/z^2,$$

or
$$[z^1+\theta(c_j-z_j^1)]/[z^2+\theta(d_j-z_j^2)]-z^1/z^2>0,$$

or
$$z^2[z^1+\theta(c_j-z_j^1)]-z^1[z^2+\theta(d_j-z_j^2)]>0.$$

(z^2 and $\bar{z}^2$ are positive, since the denominator of the objective function is positive for all feasible solutions.)

$$z^2(c_j-z_j^1)-z^1(d_j-z_j^2)>0$$

(θ being positive in the nondegenerate case if $\theta=0$, $\bar{z}=z$).

Let
$$\Delta_j=z^2(c_j-z_j^1)-z^1(d_j-z_j^2).$$

Now Δ_j is greater than zero if

Case I

$$z_j^2-d_j<0,$$
$$(z_j^1-c_j)/(z_j^2-d_j)>z^1/z^2.$$

Case II

$$z_j^2-d_j>0,$$
$$(z_j^1-c_j)/(z_j^2-d_j)<z^1/z^2.$$

Case III

$$z_j^2-d_j=0,$$
$$z_j^1-c_j<0.$$

We deduce that given a basic feasible solution $X_B=B^{-1}b$, if for any column a_j in A but not in B, $\Delta_j>0$ holds, and if at least one $u_{ij}>0$ $(i=1, 2, \cdots, m)$ then it is possible to obtain a new basic feasible solution by replacing one of the columns in B by a_j and the new value of the objective function satisfies $\bar{z} \geqq z$.

We now show that for any a_j in A not in B at least

one $\quad u_{ij}>0. \quad (i=1, 2, \cdots, m)$

If possible let all $\quad u_{ij}\leqq 0; \quad (i=1, 2, \cdots, m)$

we have the basic feasible solution

$$\textstyle\sum_{i=1}^{i=m} x_{Bi}\, b_i=b. \tag{4}$$

Suppose that we add and subtract $\bar{\theta}\, a_j$ ($\bar{\theta}$ is any scalar) to (4) we obtain

$$\textstyle\sum_{i=1}^{i=m} x_{Bi}\, b_i-\bar{\theta}\, a_j+\bar{\theta}\, a_j=b. \tag{5}$$

Since
$$-\bar{\theta}\, a_j=-\bar{\theta}\textstyle\sum_{i=1}^{i=m} u_{ij}\, b_i, \tag{6}$$

using (3) in (2) we have

$$\textstyle\sum_{i=1}^{i=m} (x_{Bi}-\bar{\theta}\, u_{ij})b_i+\bar{\theta}\, a_j=b;$$

when $\bar{\theta}>0$ we have

$$x_{Bi}-\bar{\theta}\, u_{ij}\geqq 0.$$

Since by assumption $\quad u_{ij}\leqq 0, \quad (i=1, 2, \cdots, m)$

therefore, $x_{Bi}-\bar{\theta}\, u_{ij}, \cdots, x_{Bm}-\bar{\theta}\, u_{mj}$ and $\bar{\theta}$ is a feasible solution for all $\bar{\theta}>0$. Thus the set S is unbounded contrary to our hypothesis of regularity.

In the algorithm we have shown that if we start with a basic feasible solution and if there is a vector a_j not in the basis having

$$\Delta_j>0, \tag{7}$$

then there exists another basic feasible solution such that $z \geqq z$.

If degeneracy is not present, $\bar{z}>z$. Thus we can movefrom one basis to another, changing one vector at a time so long as there is some a_j not in the basis with condition (7), and at each step z is increased.

This process cannot continue indefinitely because there are only a finite number of basis and in the absence of degeneracy no basis can ever be repeated, since z is increased at every step and the same basis cannot yield two different values of z, while at the same time the maximum is to occur at one of the basic feasible soluions.

This process will terminate only in one way, i.e., when all $\Delta_j\leqq 0$ for the columns of 'A' not in the basis.

Now for those columns of A that are in the basis

$$z_j^1=c_B'u_j=c_B'B^{-1}a_j=c_B'B^{-1}b_i=c_j,$$

and
$$z_j^2=d_B'u_j=d_B'B^{-1}a_j=d_B'B^{-1}b_i=d_j.$$

Therefore
$$\Delta_j=z^2(c_j-z_j^1)-z^1(d_j-z_j^2)=0.$$

Thus we summarize the result as follows:
Given a basic feasible solution $X_B=B^{-1}b$ with

$$z_0=(c_B'X_B+\alpha)/(d_B'X_B+\beta)$$

to the problem:

maximize $$z=(c'X+\alpha)/(d'X+\beta),$$
subject to $$AX=b,$$
$$X\geqq 0,$$

such that $\Delta_j\leqq 0$ for every column of a_j in A. Then z_0 is the maximum value of z and the basic feasible solution is an optimal solution.

TABLE I

			a_3	a_4	a_1	a_2	
			x_3	x_4	x_1	x_2	x_{Bi}/u_{ij}
$d_B\downarrow$	$c_B\downarrow$	$c_j\rightarrow$	0	0	5	3	
		$d_j\rightarrow$	0	0	5	2	
0	0	$x_3=15$	1	0	3	5	$15/3=5$
0	0	$x_4=10$	0	1	5	2	$10/5=2$
$z^2=1$	$z^1=0$	$z=0$					
		$c_j=z_j^1$	0	0	5	3	
		$d_j-z_j^2$	0	0	5	2	
		Δ_j	—	—	5	3	

TABLE II

			a_3	a_4	a_1	a_2	
			x_3	x_4	x_1	x_2	x_{Bi}/u_{ij}
$d_B\downarrow$	$c_B\downarrow$	$c_j\rightarrow$	0	0	5	3	
		$d_j\rightarrow$	0	0	5	2	
0	0	$x_3=9$	1	$-3/5$	0	$19/5$	$45/19$
5	5	$x_1=2$	0	$1/5$	1	$2/5$	5
$z^2=11$	$z^1=10$	$z=10/11$					
		$c_j-z_j^1$	0	-1	0	1	
		$d_j-z_j^2$	0	-1	0	0	
		Δ_j	—	-1	—	11	

NUMERICAL EXAMPLE

We illustrate the algorithm by solving a numerical example.

Maximize $$(5x_1+3x_2)/(5x_1+2x_2+1),$$
subject to $$3x_1+5x_2\leqq 15,$$
$$5x_1+2x_2\leqq 10,$$
$$x_1,x_2\geqq 0.$$

After adding slack variables, the problem in the standard form is

$$\begin{array}{ll} \text{maximize} & (5x_1+3x_2)/(5x_1+2x_2+1) \\ \text{subject to} & 3x_1+5x_2+x_3 \quad =15, \\ & 5x_1+2x_2 \quad +x_4=10, \\ & x_j \geqq 0. \qquad (j=1, 2, \cdots, 4) \end{array}$$

TABLE III

			a_3	a_4	a_1	a_2	
			x_3	x_4	x_1	x_2	x_{Bi}/u_{ij}
$d_B\downarrow$	$c_B\downarrow$	$c_j\rightarrow$	0	0	5	3	
		$d_j\rightarrow$	0	0	5	2	
2	3	$x_2=45/19$	$5/19$	$-3/19$	0	1	
5	5	$x_1=20/19$	$-2/19$	$5/19$	1	0	4
$z^2=209/19$	$z^1=235/19$	$z=235/209$					
		$c_j-z_j^1$	$-5/19$	$-16/19$	0	0	
		$d_j-z_j^2$	0	-1	0	0	
		Δ_j	$-1045/361$	$1121/361$	—	—	

$$A=\left\|\begin{matrix} 3 & 5 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{matrix}\right\|=(a_1, a_2, a_3, a_4).$$

Any two columns of A are linearly independent. We immediately have an initial basic feasible solution with (a_3, a_4) in the basis, the variables x_3 and x_4 have zero coefficients in the numerator and the denominator of the objective function, and thus we have

TABLE IV

			a_3	a_4	a_1	a_2	
			x_3	x_4	x_1	x_2	x_{Bi}/u_{ij}
$d_B\downarrow$	$c_B\downarrow$	$c_j\rightarrow$	0	0	5	3	
		$d_j\rightarrow$	0	0	5	2	
2	3	$x_2=3$	$1/5$	0	$3/5$	1	
0	0	$x_4=4$	$-2/5$	1	$19/5$	0	
$z^2=7$	$z^1=9$	$z=9/7$					
		$c_j-z_j^1$	$-3/5$	0	$16/5$	0	
		$d_j-z_j^2$	$-2/5$	0	$19/5$	0	
		Δ_j	$-3/5$	—	$-59/5$	—	

$$c_B=\begin{pmatrix}0\\0\end{pmatrix}, \quad d_B=\begin{pmatrix}0\\0\end{pmatrix}, \quad X_B=\begin{pmatrix}15\\10\end{pmatrix}.$$

Now let us consider Tables I–IV.

In Table I we find

$$\begin{array}{ll} \Delta_1=5, & x_1 = 0, \\ \Delta_2=3, & x_3 = 0. \end{array}$$

we choose max Δ_j (i.e., Δ_1 here). Thus z can be increased by taking a_1 into the basis. The method to determine departing variables and also the new values of u_{ij}, X_B, z_j^1, z_j^2 corresponding to new basic feasible solutions will be the same as for a linear programming problem. Thus here x_4 is a departing variable.

In Table IV all $\Delta_j \leqq 0$. Thus we have reached maximum $z = 9/7$ and the optimum solution

$$x_2 = 3, \qquad x_1 = 0,$$
$$x_4 = 4. \qquad x_3 = 0.$$

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TIMING OF CHECK OUT BEFORE A CRITICAL EVENT

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A system with constant failure rate must be operable at time T. Prior to this we permit only one check-out, and one repair if necessary. How far should we 'back off' from T in scheduling check out? Approximate solutions are derived for exponential and log-normal repair time distributions, with graphical results in the latter case.

IN RECENT years, intensive studies have been made of maintenance policies for systems whose times-to-failure and repair times are random variables. Some of these, together with a survey paper, are listed in the references. It is clear that results that hold for systems whose span of operation is finite, or that need to operate only on demand, may differ from results for systems operated in-

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