'LINEAR' PROGRAMMING WITH ABSOLUTE-VALUE FUNCTIONALS

David F. Shanno

University of Toronto, Toronto, Ontario, Canada

and

Roman L. Weil

University of Chicago, Chicago, Illinois

(Received November 24, 1969)

Consider the problem $Ax=b$; max $z=\sum c_j|x_j|$. This problem cannot, in general, be solved with the simplex method. The problem has a simplex-method solution (with unrestricted basis entry) only if c_j are nonpositive (nonnegative for minimizing problems).

CONSIDER the problem

$$\text{maximize } z=\sum c_j|x_j|, \text{ subject to } Ax=b. \tag{1}$$

Parts of the literature imply that the standard simplex method can be used to solve this problem. We point out in this note that such an implication is correct only if all c_j are nonpositive (nonnegative for minimizing problems). Otherwise the simplex method must be drastically modified. The primary application of absolute-value functionals in linear programming has been for absolute-value or l_1-metric regression analysis. Such application is always a minimization problem with all c_j equal to 1 so that the required conditions for valid use of the simplex method are met. First, we review what the literature has said; next we show an example to demonstrate the nongenerality of the implied solution; finally, we explain precisely what is happening when absolute-value functionals are used and what little can be done about the problem. [*See* Note 1.]

THE LITERATURE

ONE, PERHAPS the only, clear implication that the simplex method can be used for linear programming problems with absolute-value functionals is in a problem by HADLEY [reference 3, p. 172], which we reproduce in full:

> 5–12. Show how the simplex method can be used to solve a problem of the following type: $Ax=b$, max $z=\sum c_j|x_j|$. The variables x_j are unrestricted, and $|x_j|$ is the absolute value of x_j. Show that the same technique can be used if the cost corresponding to a negative x_j is different from that corresponding to a positive x_j. The activity vector for x_j remains, of course, the same, regardless of whether x_j is positive or negative.

Hadley presents no suggested solution, but we assume the implied method is that used by CHARNES AND COOPER[1] and by WAGNER[6] for their particular problems of l_1-metric regression. The technique, first used by CHARNES, COOPER, AND FERGUSON,[2] is to replace each unrestricted variable (e.g., x_j) in the constraints by the difference of two new nonnegative variables (e.g., $x_j^+ - x_j^-$) and to give each of the new variables the appropriate c_j in the functional. The c_j need not, as Hadley says, be the same for x_j^+ as for x_j^-.

The conditions for successful application of adjacent-extreme-point methods derived, for example, by MARTOS[5] and by Hadley [reference 4, p. 124] only *implicitly* rule out maximizing (minimizing) absolute-value functionals with positive (negative) c_j. Nowhere in the literature have we been able to find an explicit injunction against such attempts. We believed until recently that general absolute-value 'linear' programming was possible with adjacent-extreme-point methods and unrestricted basis entry.

A more general formulation of the absolute-value programming problem is

$$\text{maximize } \sum_{i=1}^{i=n} c_i |f_i(t)|, \text{ subject to } At=b,\ t \geqq 0, \tag{2}$$

where A is a $p \times m$ matrix, b a p-vector, t an m-vector, and

$$f_i(t) = \sum_{j=1}^{j=m} \alpha_{ij} t_j - \beta_i. \tag{3}$$

But this is the actual form of the l_1-regression function, where the t_i represent the regression coefficients, the c_i are the weights given to the respective data points, and β_i are the values observed for the independent variables α_{ij}. This is equivalent to the form (1) if we use the transformation of Charnes, Cooper, and Ferguson[2] and we define $x_i^+ - x_i^- = f_i(t)$, x_i^+, $x_i^- \geqq 0$.

It will be seen below that the trouble, which can arise in absolute-value linear programming, is that, in maximizing problems with positive c_j or in minimizing problems with negative c_j, there exist local optima that are not global optima.

Example.

$$\text{maximize } |x|, \text{ subject to } -4 \leqq x \leqq 2,\ x \text{ unrestricted in sign.} \tag{4}$$

Any general solution method for (1) should work for (4). In order to 'solve' this problem by the simplex method we redefine, using the notation of Charnes and Cooper [reference 1, pp. 334 ff.], $x = x^+ - x^-$, x^+, $x^- \geqq 0$; we rewrite the constraints $x \geqq -4$ as $-x^+ + x^- \leqq 4$ and $x \leqq 2$ as $x^+ - x^- \leqq 2$.

Restate problem (4) as a standard linear program:

$$\max\ (x^+ + x^-), \text{ subject to } -x^+ + x^- \leqq 4,\ x^+ - x^- \leqq 2,\ x^+,\ x^- \geqq 0. \tag{4'}$$

After appending nonnegative slack variables s_1 and s_2, we write an initial tableau:

	x^+	x^-	s_1	s_2		
c_j:	1	1	0	0		
s_1	-1	1	1	0	4	
s_2	1	-1	0	1	2	← (5)
$z_j - c_j$	-1	-1	0	0	0	

We choose, arbitrarily, to pivot in the x^+ column to obtain the second tableau:

	x^+	x^-	s_1	s_2	
s_1	0	0	1	1	6
x^+	1	−1	0	1	2
	0	−2	0	1	2

(6)

We see that we want next to pivot in the x^- column, but all its elements are nonpositive so that the introduction of x^- leads to an unbounded solution: $x^+=2+\theta$, $x^-=\theta$, for arbitrarily large positive θ is feasible with arbitrarily large objective function value $2+2\theta$.

If, instead, we had chosen to pivot in the x^- column in the initial tableau, the signal for an unbounded solution would also occur. Further, an unbounded solution will occur if the original objective function had been minimize $z=-|x|$.

MATHEMATICAL ANALYSIS OF THE PROBLEM

THE PURPOSE OF this section is to demonstrate the conditions for failure of the simplex method when applied to the absolute-value linear programming problem (*see* Note 2). Consider the problem

$$\text{maximize } \textstyle\sum_{i=1}^{i=n} c_i|x_i|, \text{ subject to } Ax=b,\ x>0, \tag{7}$$

where A is an $m\times n$ matrix, b an m-vector, x an n-vector. Let

$$x_i^+ - x_i^- \equiv x_i,\ x_i^+,\ x_i^- \geqq 0 \tag{8}$$

and note that $|x_i|=|x_i^+-x_i^-|=x_i^++x_i^-$ if and only if *not both* x_i^+ *and* x_i^- *are nonzero*. Under this restriction, we reformulate (7) as

$$\begin{aligned} &\text{maximize } \left(\textstyle\sum_{i=1}^{i=n} c_i x_i^+ + \sum_{i=1}^{i=n} c_i x_i^-\right), \text{ subject to } \hat{A}\hat{x}=b, \hat{x}\geqq 0,\\ &\text{where } \hat{A}=(A,\ -A),\ \hat{x}=(x_1^+,\cdots,x_n^+,x_1^-,\cdots,x_n^-), \text{ and}\\ &\qquad x_i^+x_i^-=0,\ i=1,\cdots,n. \end{aligned} \tag{9}$$

The restriction that causes the failure of the simplex method is the last restriction in (9), namely that $x_i^+x_i^-=0$, the restricted basis entry condition. We first show that, if $c_i<0$, this constraint is automatically handled by the simplex method. In order to demonstrate this, suppose $x_i^+>0$ for some i. Then the column a_i^+ corresponding to x_i^+ in (9) is in the basis. Since a_i^+ is in the basis, we have

$$z_i^+ - c_i = 0. \tag{10}$$

Now $c_i<0$, so that $z_i^+<0$. Since $a_i^-=-a_i^+$ (a_i^- is the column corresponding to x_i^-), we have $z_i^-=-z_i^+$, so $z_i^->0$. Also, from (10) and the fact that c_i, assumed for the moment to be negative, is the cost coefficient for both x_i^+ and x_i^-, it must be true that

$$z_i^- - c_i > 0. \tag{11}$$

Because the problem is a maximization problem, a_i will never become a candidate to enter the basis. The simplex method will converge to the proper optimum.

Suppose instead that $c_i>0$, while $x_i^+>0$ and a_i^+ again is in the basis. Repeating the above analysis, we have $z_i^+ - c_i = 0$, $-c_i < 0$, so $z_i^+ > 0$. Then $z_i^- = -z_i^+ < 0$, so that

$$z_i^- - c_i^- < 0. \tag{12}$$

Thus if x_i^+ enters the basis, x_i^- *always* becomes a candidate to enter the basis. Now since a_i^+ is in the basis and $B^{-1}a_i^+ = e_s$, where B is the basis matrix and e_s is some unit vector,

$$B^{-1}a_i^- = -e_s. \tag{13}$$

From (12) and (13), if x_i^+ is in the basis, x_i^- can always be entered in such a way as to generate an arbitrarily large value for z, and the problem has an unbounded solution. Thus when $c_i < 0$, the simplex algorithm must be modified to guarantee that the restricted basis entry condition $x_i^+ x_i^- = 0$ is satisfied. This is precisely the modification of the simplex algorithm used in separable programming problems for which convergence to a local optimum is assured; *see* Hadley [reference 4, p. 107]. Unfortunately, the local optimum found is often not a global optimum, and the problem of determining the global optimum becomes a combinatorial problem. Thus, in the example above, there are two local maxima, $x = 2$ and $x = -4$, and the one found by the simplex method depends entirely on which vector is first selected to enter the basis.

Until now we have assumed that the c_i associated with x_i^+ is the same as the c_i associated with x_i^-. As the problem quoted from Hadley suggests, x_i^+ may have a c_i^+ different from c_i^- associated with x_i^-. However, extending the above analysis it can be shown that the simplex method will converge to an absolute maximum with unrestricted basis entry only when $-c_i^- \geqq c_i^+ (-c_i^+ \geqq c_i^-$ for minimizing).

NOTES

1. We first became aware of this problem in discussions with JOHN P. GOULD. A. CHARNES, W. W. COOPER, and JOHN P. EVANS provided useful advice on an earlier version. They are in no way responsible for the existence nor, certainly, the contents of this one. Research support was provided in part by the General Electric Foundation through a grant to the Graduate School of Business of the University of Chicago, and in part by the National Science Foundation under grants to the University of Chicago.

2. One referee found an elegant proof, which used duality theory, to show that the simplex method must, in general, fail. Our presentation here is less elegant (and longer) but shows *how* the simplex method fails.

REFERENCES

1. A. CHARNES AND W. W. COOPER, *Management Models and Industrial Applications of Linear Programming*, Wiley, New York, 1961.
2. ———, ———, AND R. FERGUSON, "Optimal Estimation of Executive Compensation by Linear Programming," *Management Sci.* **1,** 307–23 (1955).
3. G. HADLEY, *Linear Programming*, Addison-Wesley, Reading, Mass., 1962.

4. ———, *Nonlinear and Dynamic Programming*, Addison-Wesley, Reading, Mass., 1964.
5. B. MARTOS, "The Direct Power of Adjacent Vertex Programming Methods," *Management Sci.* **12,** 241–252 (1965).
6. HARVEY WAGNER, "Linear Programming Techniques for Regression Analysis," *J. Am. Stat. Assoc.* **54,** 206–12 (1959).
7. ROBERT WILSON, "On Programming under Uncertainty," *Opns. Res.* **14,** 652–57 (1966).

THE OPTIMAL LOCATION OF NEW FACILITIES USING RECTANGULAR DISTANCES

G. O. Wesolowsky

McMaster University, Hamilton, Ontario, Canada

and

R. F. Love

University of Wisconsin, Madison, Wisconsin

(Received May 20, 1969)

This note describes a method for locating any number of facilities optimally in relation to any number of existing facilities. The objective is to minimize the total of load-times-distance costs in the system. Any amount of loading may be present between the new facilities and the existing facilities and between the new facilities themselves. Distances are assumed to be rectangular.

RECENT contributions to mathematical location theory include a considerable amount of work on what is often called the Steiner-Weber problem; these include articles by COOPER,[1] SEYMOUR,[2] and KUENNE AND KUHN.[3] In its simplest form, the Steiner-Weber problem concerns itself with the location of a center on a plane such that the sum of distances from that center to a number of given fixed points is minimized. This problem has been generalized in many ways. The aim of this note is to discuss some aspects of the Steiner-Weber problem under the assumption, not of Euclidean straight-line distances, but of rectangular distances. Previous work on models with rectangular distances has been done by BINDSCHEDLER AND MOORE[4] and by FRANCIS.[5,6,7]

We define the rectangular distance between two points in n-dimensional space, $(x_1, x_2, \cdots, x_n)$ and $(z_1, z_2, \cdots, z_n)$, to be

$$D_{12}=\sum_{i=1}^{i=n} |x_i-z_i|. \quad (1)$$

Figure 1 shows that the distance between points A and B in two dimensions can be measured along paths a, b, or c.

LINKED CITATIONS

References

[6] **Linear Programming Techniques for Regression Analysis**
Harvey M. Wagner
Journal of the American Statistical Association, Vol. 54, No. 285. (Mar., 1959), pp. 206-212.
Stable URL:
http://links.jstor.org/sici?sici=0162-1459%28195903%2954%3A285%3C206%3ALPTFRA%3E2.0.CO%3B2-%23