Pergamon

Appl. Math. Lett. Vol. 9, No. 3, pp. 89–96, 1996
Copyright©1996 Elsevier Science Ltd
Printed in Great Britain. All rights reserved
0893-9659/96 $15.00 + 0.00

S0893-9659(96)00038-9

Parametric Linear Semi-Infinite Programming

C.-J. LIN
Department of Industrial and Operations Engineering
The University of Michigan, Ann Arbor, MI 48109, U.S.A.

S.-C. FANG
Operations Research and Industrial Engineering
North Carolina State University, Raleigh, NC 27695-7913, U.S.A.

SOON-YI WU
Institute of Applied Mathematics, National Cheng-Kung University
Tainan, Taiwan, R.O.C.

(Received and accepted August 1995)

Abstract—By using the theory of parametric semi-infinite programming, we show that the solution of a linear semi-infinite programming problem can be obtained by solving a sequence of optimization problems with a single constraint.

Keywords—Semi-infinite programming, Linear programming, Convex programming.

1. INTRODUCTION

Many linear semi-infinite programming problems [1] appear in the following form:

$$\begin{aligned} &\max b^{\mathrm{T}} w \\ &\text{s.t.} \sum_{j=1}^{m} a_j(t) w_j \leq c(t), \qquad \forall t \in T \end{aligned} \tag{D}$$

where $b, w \in R^m$, T is a compact set in R^n, and $c(t)$, $a_j(t)$, $j = 1, \ldots, m$ are continuous functions on T.

When T is finite, say $T = \{t_1, \ldots, t_n\}$, Program (D) becomes a regular linear program with m variables and n constraints:

$$\begin{aligned} &\max b^{\mathrm{T}} w \\ &\text{s.t.} \sum_{j=1}^{m} a_{jk} w_j \leq c_k, \quad k = 1, \ldots, n \end{aligned} \tag{$\overline{\mathrm{D}}$}$$

where $a_{jk} = a_j(t_k)$ and $c_k = c(t_k)$.

A recent effort in regular linear programming [2,3] is to aggregate the n constraints into one and consider the following parameterized perturbation problem, with $p \geq 1$:

$$\begin{aligned} &\max b^{\mathrm{T}} w \\ &\text{s.t.} \log \left(\sum_{k=1}^{n} e^{p\left(\sum_{j=1}^{m} a_{jk} w_j - c_k\right)} \right)^{1/p} \leq 0. \end{aligned} \tag{$\overline{\mathrm{D}}_p$}$$

This research work is partially supported by the 1994 NCSC-Cray Research Grant.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

By observing that the p-norm approaches the max-norm as $p \to \infty$, it was shown in [2,3], $(\overline{\mathrm{D}}_p) \to (\overline{\mathrm{D}})$ in optimal solutions as $p \to \infty$. In this note, we would like to extend the approach for solving linear semi-infinite programming problems.

To simplify notations, we let $a(t) \triangleq (a_1(t), \ldots, a_m(t))^{\mathrm{T}}$, $\forall t \in T$, and $\sum_{j=1}^{m} a_j(t)w_j - c(t)$ becomes $a(t)^{\mathrm{T}}w - c(t)$.

2. MAIN RESULT

First, note that Program (D) can be rewritten as

$$\begin{aligned} &\max b^{\mathrm{T}}w \\ &\text{s.t. } f^*(w) \triangleq \max_{t\in T}\left\{\sum_{j=1}^{m} a_j(t)w_j - c(t)\right\} \le 0. \end{aligned} \tag{D'}$$

Let us consider the following perturbed problem, with $p \ge 1$:

$$\begin{aligned} &\max b^{\mathrm{T}}w \\ &\text{s.t. } f_p(w) \triangleq \log\left(\int_T e^{p\left(a(t)^{\mathrm{T}}w - c(t)\right)}\,dt\right)^{1/p} \le 0. \end{aligned} \tag{D'$_p$}$$

Our objective is to show that $(\mathrm{D}'_p) \to (\mathrm{D}')$ in optimal solutions as $p \to \infty$. To do so, we make two commonly seen assumptions:

(A1) Program (D) satisfies the Slater condition, i.e., there exists $w \in R^m$ such that $\sum_{j=1}^{m} a_j(t)w_j - c(t) < 0$, $\forall t \in T$.

(A2) The feasible domain of Program (D) is nonempty and compact.

With these two assumptions, Brosowski [4, Theorem 6 of Chapter 4] showed that if

(i) f^* is continuous;

(ii) the feasible domain $F(\mathrm{D}'_p)$ of Program (D'_p) is convex; and

(iii) on each compact set $W \subset R^m$, $f_p \to f^*$ uniformly;

then $F(\mathrm{D}'_p)$ lies in a compact set, when p is sufficiently large, and Program (D'_p) is solvable. Furthermore, if w_p^* is an optimal solution of Program (D'_p), then each accumulation point of the sequence $\{w_p^*\}$ is an optimal solution of Program (D). Therefore, w_p^* can be treated as an approximate solution of Program (D), when p is sufficiently large. We now prove the following results.

LEMMA 2.1. *The function* $f^*(w) \triangleq \max_{t\in T}\{a(t)^{\mathrm{T}}w - c(t)\}$ *is continuous.*

PROOF. For each sequence $\{w^i\} \to w$, we want to show that

$$\lim_{i\to\infty} \max_{t\in T}\{a(t)^{\mathrm{T}}w^i - c(t)\} = \max_{t\in T}\{a(t)^{\mathrm{T}}w - c(t)\}.$$

Since T is a compact set, we may assume that $\max_{t\in T}\{a(t)^{\mathrm{T}}w^i - c(t)\} = a(t_i)^{\mathrm{T}}w^i - c(t_i)$ for some $t_i \in T$, and there exists a subsequence $\{t_{i_l}\}$ converging to t^*. For each $t \in T$, we know that

$$a(t)^{\mathrm{T}}w_{i_l} - c(t) \le a(t_{i_l})^{\mathrm{T}}w_{i_l} - c(t_{i_l}).$$

As $i_l \to \infty$,

$$a(t)^{\mathrm{T}}w - c(t) \le a(t^*)^{\mathrm{T}}w - c(t^*). \tag{1}$$

Hence, $\max_{t\in T}\{a(t)^{\mathrm{T}}w - c(t)\} = a(t^*)w - c(t^*)$.

If $\lim_{i\to\infty} \max_{t\in T}\{a(t)^{\mathrm{T}}w^i - c(t)\} \ne a(t^*)^{\mathrm{T}}w - c(t^*)$, there exists $\epsilon > 0$ and a subsequence $\{t_{i_k}\}$ such that

$$\left|\left(a\left(t_{i_k}\right)^{\mathrm{T}} w_{i_k} - c\left(t_{i_k}\right)\right) - \left(a\left(t^*\right)^{\mathrm{T}} w - c\left(t^*\right)\right)\right| > \epsilon, \quad \text{for } i_k \text{ being sufficiently large,} \tag{2}$$

and there is a subsequence $\{t_{i_{k_j}}\}$ of $\{t_{i_k}\}$ which converges to a point $\bar{t} \in T$.

By equation (1), $a(\bar{t})^{\mathrm{T}}w - c(\bar{t}) \leq a(t^*)^{\mathrm{T}}w - c(t^*)$. Again, for each $t \in T$, $a(t)^{\mathrm{T}}w_{i_{k_j}} - c(t) \leq a(t_{i_{k_j}})^{\mathrm{T}}w_{i_{k_j}} - c(t_{i_{k_j}})$. As $i_{k_j} \to \infty$, $a(t)^{\mathrm{T}}w - c(t) \leq a(\bar{t})^{\mathrm{T}}w - c(\bar{t})$. Hence,

$$a(\bar{t})^{\mathrm{T}}w - c(\bar{t}) = a(t^*)^{\mathrm{T}}w - c(t^*)$$

which contradicts equation (2) and completes the proof. ∎

LEMMA 2.2. *The set* $F(D_p') \triangleq \{w \in R^m \mid \log(\int_T e^{p(a(t)^{\mathrm{T}}w - c(t))}\,dt)^{1/p} \leq 0\}$ *is convex.*

PROOF. Given w^1 and w^2 in $F(\mathrm{D}_p')$, we would like to prove that

$$\log\left(\int_T e^{p\left(a(t)^{\mathrm{T}}\left(\lambda w^1 + (1-\lambda)w^2\right) - c(t)\right)}\,dt\right)^{1/p} \leq 0, \qquad \text{for } 0 \leq \lambda \leq 1.$$

This is equivalent to showing

$$\int_T e^{p\left(a(t)^{\mathrm{T}}\left(\lambda w^1 + (1-\lambda)w^2\right) - c(t)\right)}\,dt \leq 1, \qquad \text{for } 0 \leq \lambda \leq 1.$$

By the fact that the exponential function is convex, we have

$$\begin{aligned}\int_T e^{p\left(a(t)^{\mathrm{T}}\left(\lambda w^1 + (1-\lambda)w^2\right) - c(t)\right)}\,dt &= \int_T e^{p\lambda\left(a(t)^{\mathrm{T}}w^1 - c(t)\right) + p(1-\lambda)\left(a(t)^{\mathrm{T}}w^2 - c(t)\right)}\,dt \\ &\leq \int_T \lambda e^{p\left(a(t)^{\mathrm{T}}w^1 - c(t)\right)}\,dt + \int_T (1-\lambda)e^{p\left(a(t)^{\mathrm{T}}w^2 - c(t)\right)}\,dt \\ &= \lambda\int_T e^{p\left(a(t)^{\mathrm{T}}w^1 - c(t)\right)}\,dt + (1-\lambda)\int_T e^{p\left(a(t)^{\mathrm{T}}w^2 - c(t)\right)}\,dt \leq 1.\end{aligned}$$ ∎

The following two lemmas will lead to the uniform convergence of $f_p \to f^*$.

LEMMA 2.3. *For any compact subset* $W \subset R^m$, *and* $0 < l < 1$, *there exists a sufficiently large* $P > 0$ *such that for each integer* $p > P$, *there exists a partition* $\{T_1, \ldots, T_p\}$ *of* T *satisfying*

$$\frac{\max_{j=1,\ldots,p} \min_{t \in T_j} e^{a(t)^{\mathrm{T}}w - c(t)}}{l \cdot \max_{t \in T} e^{a(t)^{\mathrm{T}}w - c(t)}} > 1, \qquad \forall w \in W.$$

PROOF. If our claim is false, then there exists a compact subset $W \subset R^m$, a constant $0 < l < 1$, a convergent sequence $\{w^i\} \to w^*$, and for each i, there corresponds a p_i with $p_i \to \infty$ as $i \to \infty$, such that

$$\frac{\max_{j=1,\ldots,p_i} \min_{t \in T_j} e^{a(t)^{\mathrm{T}}w^i - c(t)}}{l \cdot \max_{t \in T} e^{a(t)^{\mathrm{T}}w^i - c(t)}} \leq 1, \text{ for every partition } \{T_1, T_2, \ldots, T_{p_i}\} \text{ of } T \text{ and } i \to \infty. \quad (3)$$

For each $t \in T$, by Taylor expansion, we have

$$e^{a(t)^{\mathrm{T}}w^i - c(t)} = e^{a(t)^{\mathrm{T}}w^* - c(t)} + e^{a(t)^{\mathrm{T}}\hat{w}^i - c(t)}\left(a(t)^{\mathrm{T}}\left(w^i - w^*\right)\right)$$

where $\hat{w}^i$ lies between w^i and w^*.

Since $w^i \to w^*$ as $i \to \infty$, there exists a closed ball $B(w^*, R)$ such that $w^i, \hat{w}^i$ are in $B(w^*, R)$, when i is sufficiently large. Since T is compact, there exists $M > 0$ such that, for each T_j, $j = 1, \ldots, p_i$,

$$\min_{t \in T_j} e^{a(t)^{\mathrm{T}}w^* - c(t)} - M\left\|w^i - w^*\right\| \leq \min_{t \in T_j} e^{a(t)^{\mathrm{T}}w^i - c(t)} \leq \min_{t \in T_j} e^{a(t)^{\mathrm{T}}w^* - c(t)} + M\left\|w^i - w^*\right\|.$$

Hence,

$$\lim_{p_i\to\infty}\max_{j=1,\dots,p_i}\min_{t\in T_j} e^{a(t)^{\mathrm{T}}w^i-c(t)} \le \lim_{p_i\to\infty}\left(\max_{j=1,\dots,p_i}\min_{t\in T_j} e^{a(t)^{\mathrm{T}}w^*-c(t)} + M\left\|w^i-w^*\right\|\right)$$

and

$$\lim_{p_i\to\infty}\max_{j=1,\dots,p_i}\min_{t\in T_j} e^{a(t)^{\mathrm{T}}w^i-c(t)} \ge \lim_{p_i\to\infty}\left(\max_{j=1,\dots,p_i}\min_{t\in T_j} e^{a(t)^{\mathrm{T}}w^*-c(t)} - M\left\|w^i-w^*\right\|\right).$$

Therefore, if we take a particular partition $\{T_1,\dots,T_{p_i}\}$ of T such that $\max_{j=1,\dots,p_i} d(T_j)\to 0$, as $p_i\to\infty$, where $d(T_j)$ is the largest diameter of T_j, then

$$\lim_{p_i\to\infty}\max_{j=1,\dots,p_i}\min_{t\in T_j} e^{a(t)^{\mathrm{T}}w^i-c(t)} = \max_{t\in T} e^{a(t)^{\mathrm{T}}w^*-c(t)}, \tag{4}$$

and

$$\lim_{p_i\to\infty} l\cdot\max_{t\in T} e^{a(t)^{\mathrm{T}}w^i-c(t)} = l\cdot\max_{t\in T} e^{a(t)^{\mathrm{T}}w^*-c(t)}. \tag{5}$$

From equations (4) and (5), we have

$$\lim_{p_i\to\infty}\frac{\max_{j=1,\dots,p_i}\min_{t\in T_j} e^{a(t)^{\mathrm{T}}w^i-c(t)}}{l\cdot\max_t e^{a(t)^{\mathrm{T}}w^i-c(t)}} = \frac{1}{l} > 1.$$

This contradicts equation (3) and completes the proof. ∎

LEMMA 2.4. *For any compact subset $W\subset R^m$, f_p converges to f^* uniformly on W.*

PROOF. By the definition of Riemann integral, if T is divided into p sections, then

$$\begin{aligned}
&\left(\int_T e^{p\left(a(t)^{\mathrm{T}}w-c(t)\right)}\,dt\right)^{1/p}\\
&\le\left(\frac{\mathrm{Vol}(T)}{p}\left(\max_{t\in T_1} e^{p\left(a(t)^{\mathrm{T}}w-c(t)\right)}+\cdots+\max_{t\in T_p} e^{p\left(a(t)^{\mathrm{T}}w-c(t)\right)}\right)\right)^{1/p}\\
&=\left(\frac{\mathrm{Vol}(T)}{p}\left(\frac{\max_{t\in T_1} e^{p\left(a(t)^{\mathrm{T}}w-c(t)\right)}}{\max_{t\in T} e^{p\left(a(t)^{\mathrm{T}}w-c(t)\right)}}+\cdots+\frac{\max_{t\in T_p} e^{p\left(a(t)^{\mathrm{T}}w-c(t)\right)}}{\max_{t\in T} e^{p\left(a(t)^{\mathrm{T}}w-c(t)\right)}}\right)\right)^{1/p}\left(\max_{t\in T} e^{a(t)^{\mathrm{T}}w-c(t)}\right)\\
&\le\left(\frac{\mathrm{Vol}(T)}{p}p\right)^{1/p}\max_{t\in T} e^{a(t)^{\mathrm{T}}w-c(t)}\\
&=(\mathrm{Vol}(T))^{1/p}\max_{t\in T} e^{a(t)^{\mathrm{T}}w-c(t)}.
\end{aligned} \tag{6}$$

By Lemma 2.3, for $0<l<1$, there exists $P>0$, such that for $p>P$,

$$\begin{aligned}
&\left(\int_T e^{p\left(a(t)^{\mathrm{T}}w-c(t)\right)}\,dt\right)^{1/p}\\
&\ge\left(\frac{\mathrm{Vol}(T)}{p}\left(\min_{t\in T_1} e^{p\left(a(t)^{\mathrm{T}}w-c(t)\right)}+\cdots+\min_{t\in T_p} e^{p\left(a(t)^{\mathrm{T}}w-c(t)\right)}\right)\right)^{1/p}\\
&=\left(\frac{\mathrm{Vol}(T)}{p}\left(\frac{\min_{t\in T_1} e^{p\left(a(t)^{\mathrm{T}}w-c(t)\right)}}{\max_{t\in T} e^{p\left(a(t)^{\mathrm{T}}w-c(t)\right)}}+\cdots+\frac{\min_{t\in T_p} e^{p\left(a(t)^{\mathrm{T}}w-c(t)\right)}}{\max_{t\in T} e^{p\left(a(t)^{\mathrm{T}}w-c(t)\right)}}\right)\right)^{1/p}\left(\max_{t\in T} e^{a(t)^{\mathrm{T}}w-c(t)}\right)\\
&\ge\left(\frac{\mathrm{Vol}(T)}{p}\left(\frac{\max_{j=1,\dots,p}\min_{t\in T_j} e^{\left(a(t)^{\mathrm{T}}w-c(t)\right)}}{l\cdot\max_{t\in T} e^{a(t)^{\mathrm{T}}w-c(t)}}\right)^{p}\right)^{1/p}\left(l\cdot\max_{t\in T} e^{a(t)^{\mathrm{T}}w-c(t)}\right)\\
&\ge\left(\frac{\mathrm{Vol}(T)}{p}\right)^{1/p}\left(l\cdot\max_{t\in T} e^{a(t)^{\mathrm{T}}w-c(t)}\right).
\end{aligned} \tag{7}$$

This means that, for $0 < l < 1$, there exists a sufficiently large P such that for $p > P$,

$$\left[\left(\frac{\text{Vol}(T)}{p}\right)^{1/p} l - 1\right] \max_{t\in T} e^{a(t)^{\mathrm{T}}w - c(t)} \leq \left(\int_T e^{p\left(a(t)^{\mathrm{T}}-c(t)\right)}\right)^{1/p} - \max_{t\in T} e^{a(t)^{\mathrm{T}}w-c(t)}$$
$$\leq \left[(\text{Vol}(T))^{1/p} - 1\right] \max_{t\in T} e^{a(t)^{\mathrm{T}}w-c(t)}.$$

Since $\max_{t\in T} e^{a(t)^{\mathrm{T}}w-c(t)}$ is bounded on W, for any $\epsilon > 0$, there exists $0 < \bar{l} < 1$ such that there exists a sufficiently large P and for $p > P$,

$$\begin{aligned}
-\epsilon &\leq \left[\left(\frac{\text{Vol}(T)}{p}\right)^{1/p} \bar{l} - 1\right] \max_{t\in T} e^{a(t)^{\mathrm{T}}w-c(t)} \\
&\leq \left(\int_T e^{p\left(a(t)^{\mathrm{T}}-c(t)\right)}\right)^{1/p} - \max_{t\in T} e^{a(t)^{\mathrm{T}}w-c(t)} \\
&\leq \left[(\text{Vol}(T))^{1/p} - 1\right] \max_{t\in T} e^{a(t)^{\mathrm{T}}w-c(t)} \\
&\leq \epsilon.
\end{aligned}$$

Hence, for any $\epsilon > 0$, there exists P, such that for $p > P$ and $w \in W$, we have

$$\left|\left(\int_T e^{p\left(a(t)^{\mathrm{T}}w-c(t)\right)}\, dt\right)^{1/p} - \max_{t\in T} e^{a(t)^{\mathrm{T}}w-c(t)}\right| < \epsilon.$$

This means that $(\int_T e^{p(a(t)^{\mathrm{T}}w-c(t)}\, dt)^{1/p}$ converges to $\max_{t\in T}\{e^{a(t)^{\mathrm{T}}w-c(t)}\}$ uniformly on W. Furthermore, by using [5, Exercise 9.4, p. 248], we know that $\log(\int_T e^{p(a(t)^{\mathrm{T}}w-c(t)}\, dt)^{1/p}$ converges to $\max_{t\in T}\{a(t)^{\mathrm{T}}w - c(t)\}$ uniformly on W. ∎

Summarizing Lemmas 2.1, 2.2 and 2.4, we can conclude that $F(\mathrm{D}'_p)$ lies in a compact set, when p is sufficient large, and Program (D'_p) is solvable. Furthermore, if w_p^* is an optimal solution of Program (D'_p), then each accumulation point of the sequence $\{w_p^*\}$ is an optimal solution of Program (D').

3. EXTENSIONS

The concept of aggregating infinitely many constraints into one single constraint does not have to be limited to $f^*(w) = \max_{t\in T}\{a(t)^{\mathrm{T}}w - c(t)\} \leq 0$. In fact, we can rewrite Program (D) as

$$\begin{aligned}
&\max b^{\mathrm{T}}w \\
&\text{s.t. } \bar{f}^*(w) \triangleq \int_{\{t\in T | a(t)^{\mathrm{T}}w-c(t)\geq 0\}} \left(a(t)^{\mathrm{T}}w - c(t)\right)^2 dt \leq 0.
\end{aligned} \tag{D''}$$

In this setting, we can define Program (D''_p) in the following form:

$$\begin{aligned}
&\max b^{\mathrm{T}}w \\
&\text{s.t. } \bar{f}_p(w) \triangleq \left\{\int_{\{t\in T | a(t)^{\mathrm{T}}w-c(t)\geq 0\}} \left(a(t)^{\mathrm{T}}w - c(t)\right)^2 dt - \frac{1}{p}\right\} \leq 0.
\end{aligned} \tag{D''_p}$$

Note that for any $w \in R^m$, $(a(t)^{\mathrm{T}}w - c(t))^2$ is a bounded measurable function on t and $\{t \in T \mid a(t)^{\mathrm{T}}w - c(t) \geq 0\}$ is a measurable set. Hence for any $w \in R^m$, the Lebesgue integral $\int_{\{t\in T | a(t)^{\mathrm{T}}w-c(t)\geq 0\}} (a(t)^{\mathrm{T}}w - c(t))^2\, dt$ exists and both Programs (D'') and (D''_p) are well defined.

Similarly we can show that $(\mathrm{D}''_p) \to (\mathrm{D}'')$ in optimal solutions, as $p \to \infty$, by showing the previously mentioned three conditions.

LEMMA 3.1. *$\bar{f}^*(w)$ and $\bar{f}_p(w)$ are continuous functions.*

PROOF. In this proof, we denote $a(t)^{\mathrm{T}}w - c(t)$ by $g(w,t)$. For each sequence $\{w^i\} \to w$, we first show that

$$\lim_{i\to\infty} \int_{\{t\in T | g(w^i,t)\geq 0\}} (g(w^i,t))^2\,dt = \int_{\{t\in T | g(w,t)\geq 0\}} (g(w,t))^2\,dt. \tag{8}$$

For each w^i, we partition the set $\{t \in T \mid g(w^i,t) \geq 0\} \cup \{t \in T \mid g(w,t) \geq 0\}$ into $T_1^i \cup T_2^i \cup T_3^i$, where

$$\begin{aligned} T_1^i &= \{t \mid g(w^i,t) \geq 0 \text{ and } g(w,t) \geq 0\}, \\ T_2^i &= \{t \mid g(w^i,t) \geq 0 \text{ and } g(w,t) < 0\}, \qquad \text{and} \\ T_3^i &= \{t \mid g(w^i,t) < 0 \text{ and } g(w,t) \geq 0\}. \end{aligned}$$

Hence,

$$\begin{aligned} &\int_{\{t\in T | g(w^i,t)\geq 0\}} \left(g(w^i,t)\right)^2\,dt - \int_{\{t\in T | g(w,t)\geq 0\}} (g(w,t))^2\,dt \\ &\qquad = \int_{T_1^i} \left(\left(g\left(w^i,t\right)\right)^2 - (g(w,t))^2\right)\,dt + \int_{T_2^i} \left(g\left(w^i,t\right)\right)^2\,dt - \int_{T_3^i} (g(w,t))^2\,dt. \end{aligned}$$

Since $g(w,t)$ is continuous in w and t, and T is compact, $g(w^i,t) \to g(w,t)$ uniformly on T. Therefore, for $\epsilon > 0$, there exists $I_1 > 0$ such that for $i \geq I_1$, $|(g(w^i,t))^2 - (g(w,t))^2| \leq \epsilon/3\mathrm{Vol}(T)$, $\forall t \in T$, and then $|\int_{T_1^i} ((g(w^i,t))^2 - (g(w,t))^2)\,dt| \leq \epsilon/3$. Also, there exists $I_2 > 0$ such that for $i \geq I_2$,

$$\begin{aligned} &\left|g\left(w^i,t\right) - g(w,t)\right| \leq \sqrt{\frac{\epsilon}{3\mathrm{Vol}(T)}}, \qquad \forall t \in T, \qquad \text{and then,} \\ &\qquad 0 \leq g(w^i,t) \leq \sqrt{\frac{\epsilon}{3\mathrm{Vol}(T)}} + g(w,t) \leq \sqrt{\frac{\epsilon}{3\mathrm{Vol}(T)}}, \qquad \forall t \in T_2^i. \end{aligned}$$

Similarly, for $i \geq I_2$,

$$\begin{aligned} &\left|g\left(w^i,t\right) - g(w,t)\right| \leq \sqrt{\frac{\epsilon}{3\mathrm{Vol}(T)}}, \qquad \forall t \in T, \text{ and then,} \\ &\qquad 0 \leq g(w,t) \leq \sqrt{\frac{\epsilon}{3\mathrm{Vol}(T)}} + g\left(w^i,t\right) \leq \sqrt{\frac{\epsilon}{3\mathrm{Vol}(T)}}, \qquad \forall t \in T_3^i. \end{aligned}$$

Hence, for $i \geq I_2$,

$$\left|\int_{T_2^i} ((g(w^i,t))^2\,dt\right| \leq \frac{\epsilon}{3} \quad \text{and} \quad \left|\int_{T_3^i} (g(w,t))^2)\,dt\right| \leq \frac{\epsilon}{3}.$$

Therefore, for $\epsilon > 0$, we can select $I = \max\{I_1, I_2\}$ such that, for $i \geq I$,

$$\begin{aligned} &\left|\int_{\{t\in T | g(w^i,t)\geq 0\}} \left(g\left(w^i,t\right)\right)^2\,dt - \int_{\{t\in T | g(w,t)\geq 0\}} (g(w,t))^2\,dt\right| \\ &\qquad \leq \left|\int_{T_1^i} \left(\left(g\left(w^i,t\right)\right)^2 - (g(w,t))^2\right)\,dt\right| + \left|\int_{T_2^i} \left(g\left(w^i,t\right)\right)^2\,dt\right| + \left|\int_{T_3^i} (g(w,t))^2\,dt\right| \leq \epsilon. \end{aligned}$$

Then equation (8) is proven and we know that $\lim_{i\to\infty} \bar{f}^*(w^i) = \bar{f}^*(w)$ and $\lim_{i\to\infty} \bar{f}_p(w^i) = \bar{f}_p(w)$. ∎

LEMMA 3.2. *The set* $F(D_p'') \triangleq \{w \in R^m \mid \int_{\{t|a(t)^T w - c(t) \geq 0\}} (a(t)^T w - c(t))^2\, dt - 1/p \leq 0\}$ *is convex.*

PROOF. Similar to the proof of Lemma 3.1, we denote $a(t)^T w - c(t)$ by $g(w,t)$. For each $w^1, w^2 \in F(D_p'')$, we would like to show that $\lambda w^1 + (1-\lambda)w^2 \in F(D_p'')$, for $0 \leq \lambda \leq 1$.

Since $g(w,t)$ is convex in w, we have

$$0 \leq g(\lambda w^1 + (1-\lambda)w^2, t) \leq \lambda g(w^1,t) + (1-\lambda) g(w^2,t),$$

for each $t \in T$ such that $g(\lambda w^1 + (1-\lambda)w^2, t) \geq 0$. Hence, there are three possible cases: $g(w^1,t) \geq 0$ and $g(w^2,t) \geq 0$; $g(w^1,t) \geq 0$ and $g(w^2,t) < 0$; and $g(w^1,t) < 0$ and $g(w^2,t) \geq 0$.

We can partition the set $\{t \in T \mid g(\lambda w^1 + (1-\lambda)w^2, t) \geq 0\}$ into $T_1 \cup T_2 \cup T_3$, where

$$T_1 = \left\{t \mid g\left(\lambda w^1 + (1-\lambda)w^2, t\right) \geq 0,\ g\left(w^1, t\right) \geq 0 \text{ and } g\left(w^2, t\right) \geq 0\right\},$$
$$T_2 = \left\{t \mid g\left(\lambda w^1 + (1-\lambda)w^2, t\right) \geq 0,\ g\left(w^1, t\right) \geq 0 \text{ and } g\left(w^2, t\right) < 0\right\}, \quad \text{and}$$
$$T_3 = \left\{t \mid g\left(\lambda w^1 + (1-\lambda)w^2, t\right) \geq 0,\ g\left(w^1, t\right) < 0 \text{ and } g\left(w^2, t\right) \geq 0\right\}.$$

Then

$$\begin{aligned}
&\int_{\{t|g(\lambda w^1+(1-\lambda)w^2,t)\geq 0\}} g\left(\lambda w^1 + (1-\lambda)w^2, t\right)^2\, dt \\
&= \int_{T_1} g\left(\lambda w^1 + (1-\lambda)w^2, t\right)^2\, dt + \int_{T_2} g\left(\lambda w^1 + (1-\lambda)w^2, t\right)^2\, dt + \int_{T_3} g\left(\lambda w^1 + (1-\lambda)w^2, t\right)^2\, dt \\
&\leq \int_{T_1} \left(\lambda g\left(w^1, t\right) + (1-\lambda) g\left(w^2, t\right)\right)^2\, dt + \int_{T_2} \left(\lambda g\left(w^1, t\right) + (1-\lambda) g\left(w^2, t\right)\right)^2\, dt \\
&\quad + \int_{T_3} \left(\lambda g\left(w^1, t\right) + (1-\lambda) g\left(w^2, t\right)\right)^2\, dt \\
&\leq \int_{T_1} \left(\lambda g\left(w^1, t\right) + (1-\lambda) g\left(w^2, t\right)\right)^2\, dt + \int_{T_2} \left(\lambda g\left(w^1, t\right)\right)^2\, dt + \int_{T_3} \left((1-\lambda) g\left(w^2, t\right)\right)^2\, dt \\
&= \int_{T_1} \left(\lambda^2 g\left(w^1, t\right)^2 + 2\lambda(1-\lambda) g\left(w^1, t\right) g\left(w^2, t\right) + (1-\lambda)^2 g\left(w^2, t\right)^2\right) dt + \int_{T_2} \lambda^2 g\left(w^1, t\right)^2\, dt \\
&\quad + \int_{T_3} (1-\lambda)^2 g\left(w^2, t\right)^2\, dt \\
&\leq \int_{T_1} 2\lambda(1-\lambda) g\left(w^1, t\right) g\left(w^2, t\right)\, dt + \lambda^2 \int_{\{t|g(w^1,t)\geq 0\}} g\left(w^1, t\right)^2\, dt + (1-\lambda)^2 \int_{\{t|g(w^2,t)\geq 0\}} g\left(w^2, t\right)^2\, dt \\
&= \int_{T_1} 2\sqrt{\lambda(1-\lambda)}\sqrt{(1-\lambda) - (1-\lambda)^2} g\left(w^1, t\right) g\left(w^2, t\right)\, dt + \lambda^2 \int_{\{t|g(w^1,t)\geq 0\}} g\left(w^1, t\right)^2\, dt \\
&\quad + (1-\lambda)^2 \int_{\{t|g(w^2,t)\geq 0\}} g\left(w^2, t\right)^2\, dt \\
&\leq \int_{T_1} \lambda(1-\lambda) g\left(w^1, t\right)^2\, dt + \int_{T_1} \left((1-\lambda) - (1-\lambda)^2\right) g\left(w^2, t\right)^2\, dt \\
&\quad + \lambda^2 \int_{\{t|g(w^1,t)\geq 0\}} g\left(w^1, t\right)^2\, dt + (1-\lambda)^2 \int_{\{t|g(w^2,t)\geq 0\}} g\left(w^2, t\right)^2\, dt \\
&\leq (\lambda - \lambda^2) \int_{\{t|g(w^1,t)\geq 0\}} g\left(w^1, t\right)^2\, dt + \left((1-\lambda) - (1-\lambda)^2\right) \int_{\{t|g(w^2,t)\geq 0\}} g\left(w^2, t\right)^2\, dt \\
&\quad + \lambda^2 \int_{\{t|g(w^1,t)\geq 0\}} g\left(w^1, t\right)^2\, dt + (1-\lambda)^2 \int_{\{t|g(w^2,t)\geq 0\}} g\left(w^2, t\right)^2\, dt \\
&\leq \lambda \int_{\{t|g(w^1,t)\geq 0\}} g\left(w^1, t\right)^2\, dt + (1-\lambda) \int_{\{t|g(w^2,t)\geq 0\}} g\left(w^2, t\right)^2\, dt \\
&\leq \frac{1}{p}.
\end{aligned}$$

Hence, $\lambda w^1 + (1-\lambda)w^2 \in F(D_p'')$ and $F(D_p'')$ is convex. ∎

LEMMA 3.3. *For any compact subset* W *in* R^m, $\bar{f}_p(w) \to \bar{f}^*(w)$ *uniformly on* W.

PROOF. Since $\bar{f}_p(w) - \bar{f}^*(w) = 1/p$, for any compact set W and $\epsilon > 0$, we have a sufficiently large $P > 0$ such that, for $p > P, w \in W$, $|\bar{f}_p(w) - \bar{f}^*(w)| = 1/p < \epsilon$. This proves the lemma. ∎

The computational aspects of using different aggregation schemes are subject to further investigation.

REFERENCES

1. R. Hettich and K. Kortanek, Semi-infinite programming: Theory, method and applications, *SIAM Review* **35**, 380–429 (1993).
2. S.-C. Fang and H.-S.J.Tsao, A dual perturbation view of linear programming, Research Report UCB-ITS-RR-94-6, Institute of Transportation Studies, University of California, Berkeley, CA, (1994), (to appear in *Zeitschrift für Operations Research*).
3. S.-C. Fang and H.-S.J.Tsao, Perturbing dual feasible region for solving convex quadratic programs, OR Report 298, North Carolina State University, Raleigh, NC, (November 1994).
4. B. Brosowski, *Parametric Semi-infinite Optimization*, Verlag Peter Lang, Frankfurt, Germany, (1982).
5. T.M. Apostol, *Mathematical Analysis*, Addison-Wesley, Redwood City, CA, (1977).