Totally Unimodular Matrices

Lecture Notes: ISE/OR/MA 766

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Let $A$ be an $m \times n$ integral matrix with full row rank and $b$ an $m \times 1$ integral vector.

\textbf{LP:} \quad \min \{ c^T x : Ax = b, \ x \geq 0 \}

\textbf{IP:} \quad \min \{ c^T x : Ax = b, \ x \in \mathbb{Z}_+^n \}

- Fundamental Theory of LP.
- Basic solution: $x = (x_B, x_N) = (B^{-1}b, 0)$
**Observation:** If the optimal basis $B^*$ has $\det B^* = \pm 1$, the optimal basic solution $x^* = (B^*)^{-1}b$ is integral.

**Why?** Cramer’s rule. $B^{-1} = \frac{B^\alpha}{\det B}$

**Question:** Under what conditions do all bases satisfy $\det(B) = \pm 1$?
**Definition:** A square integral matrix $B$ is unimodular (UM) if $\det B = \pm 1$.

**Definition:** An integral matrix $A$ is totally unimodular (TUM) if every square nonsingular submatrix of $A$ is UM.

**Observation:** If $A$ is TUM, then $a_{ij} \in \{-1, 0, 1\}$. 
**Proposition:** Let $A$ be a TUM matrix. Multiplying any row or column of $A$ by $-1$ results in a TUM matrix.

**Proposition:** Let $A$ be a TUM matrix. Then the following matrices are all TUM:

$$-A, \; A^T, \; [A, I], \; [A, -A].$$
**Definition:** A polyhedron is integral if every extreme point is integral.

**Proposition:** Let \( A \) be an \( m \times n \) integral TUM matrix. the following polyhedrons are all integral for any vectors \( b \) and \( u \) of integers:

\[
\begin{align*}
\{ x \in \mathbb{R}^n : & \ Ax \leq b \} \\
\{ x \in \mathbb{R}^n : & \ Ax \geq b \} \\
\{ x \in \mathbb{R}^n : & \ Ax \leq b, \ x \geq 0 \} \\
\{ x \in \mathbb{R}^n : & \ Ax = b, \ x \geq 0 \} \\
\{ x \in \mathbb{R}^n : & \ Ax = b, \ 0 \leq x \leq u \}
\end{align*}
\]
**Theorem:** If $A$ is an $m \times n$ integral matrix with full row rank, the following are equivalent:

- Every basis $B$ is UM, i.e., $\det B = \pm 1$.
- The extreme points of $\{x \in R^n : Ax = b, \ x \geq 0\}$ are integral for all integral vectors $b$.
- Every basis has an integral inverse.
**Corollary:** If $A$ is an $m \times n$ integral matrix, the following are equivalent:

- $A$ is TUM.
- The extreme points of $\{x \in \mathbb{R}^n : Ax \leq b, \ x \geq 0\}$ are integral for all integral vectors $b$.
- Every nonsingular submatrix of $A$ has an integral inverse.

- Hoffman and Kruskal (1956)
• A linear programming problem with a totally unimodular coefficient matrix yields an optimal solution in integers for any objective vector and any integer vector on the right-hand side of the constraints.

• There are non-unimodular problems which yield integral optimal solutions for any objective vector but only certain integer constraint vectors. (Chapter 6–8, Eugene Lawler’s Book)

• There are non-unimodular problems which yield integral optimal solutions for any integer constraint vector but only certain objective vectors. (Page 165–168, Eugene Lawler’s Book)
**Question:** Given a matrix $A$, how do we know it is totally unimodular or not?

Matrices that are not TUM:

\[
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}
\quad
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{pmatrix}
\]
Totally Unimodular Matrices

Matrices that are TUM:

\[
\begin{pmatrix}
1 & -1 & 0 & 0 & -1 \\
-1 & 1 & -1 & 0 & 0 \\
0 & -1 & 1 & -1 & 0 \\
0 & 0 & -1 & 1 & -1 \\
-1 & 0 & 0 & -1 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & -1 & 0 & 0 & -1 \\
-1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]
• There do not seem to be any easily tested necessary and sufficient conditions for total unimodularity.

• There exist some characterization theorems for totally unimodular matrices. (Ghouila-Houri (1962) and Camion (1965))

• There is also an easily tested set of sufficient (but not necessary) conditions for total unimodularity.
Camion’s Characterization

**Definition:** A matrix $A$ is Eulerian if the sum of the elements in each row and each column is even.

**Theorem:** A $(0, +1, -1)$ matrix $A$ is totally unimodular if and only if the sum of the elements in each Eulerian square submatrix is a multiple of 4.

$\sqrt{\text{Camion (1963a, 1963b, 1965)}}$
Eulerian Matrices that are not TUM:

\[
\begin{pmatrix}
1 & 0 & -1 \\
1 & -1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\quad
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{pmatrix}
\]
Ghouila-Houri’s Characterization

**Theorem:** An $m \times n$ integral matrix $A$ is totally unimodular if and only if for each set $R \subseteq \{1, 2, \cdots, m\}$ can be divided into two disjoint sets $R_1$ and $R_2$ such that

$$\sum_{i \in R_1} a_{ij} - \sum_{i \in R_2} a_{ij} \in \{-1, 0, 1\}, \quad j = 1, 2, \cdots, n$$

√ Ghouila-Houri (1962), Berge (1973) and Commoner (1973)

√ Tamir (1976): a short proof based on Camion’s theorem.
**Theorem:** A $(0, +1, -1)$ matrix $A$ is totally unimodular if both of the following conditions are satisfied:

- Each column contains at most two nonzero elements.
- The rows of $A$ can be partitioned into two sets $A_1$ and $A_2$ such that two nonzero entries in a column are in the same set of rows if they have different signs and in different sets of rows if they have the same sign.

**Corollary:** A $(0, +1, -1)$ matrix $A$ is totally unimodular if it contains no more than one $+1$ and no more than one $-1$ in each column.
TUM matrices

\[
\begin{pmatrix}
1 & -1 & -1 & 0 \\
-1 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & -1 & 0 & 0 & -1 \\
-1 & 1 & -1 & 0 & 0 \\
0 & -1 & 1 & -1 & 0 \\
0 & 0 & -1 & 1 & -1 \\
-1 & 0 & 0 & -1 & 1
\end{pmatrix}
\]
**Definition:** A \((0, +1)\) matrix \(A\) has the consecutive one’s property if for any column \(j\), \(a_{ij} = a_{i'j} = 1\) with \(i < i'\) implies \(a_{lj} = 1\) for \(i < l < i'\).

**Corollary:** A matrix with the consecutive one’s property is TUM.

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
Theorem: The node-arc incidence matrix of a directed graph is TUM.

Why? Exactly one $1$ and one $-1$ in each column.

Integral Circulation Theorem: For the minimum cost circulation problem, if all lower bounds and capacities are integers and there exists a finite optimal circulation, then there exists an integral optimal circulation (whether or not arc costs are integers).
Minimum Cost Circulation Problems

\[
\begin{array}{l}
\text{min } \sum_{i,j} a_{ij} x_{ij} \\
\text{s.t. } \\
\sum_j x_{ji} - \sum_i x_{ij} = 0, \quad \forall \ i, \\
0 \leq l_{ij} \leq x_{ij} \leq c_{ij}, \quad \forall \ i, \ j.
\end{array}
\]

Introducing the slack variables:

\[
\begin{align*}
-x_{ij} + r_{ij} &= -l_{ij} \\
-x_{ij} + s_{ij} &= c_{ij}
\end{align*}
\]
Minimum Cost Circulation Problems

\[
\begin{align*}
\min & \ a^T x \\
\text{s.t.} & \ A(x, r, s) = b, \\
& \ x, r, s \geq 0.
\end{align*}
\]

\[
A = \begin{pmatrix}
G & 0 & 0 \\
-I_m & I_m & 0 \\
I_m & 0 & I_m
\end{pmatrix} \quad b = \begin{pmatrix}
0 \\
-l \\
c
\end{pmatrix}
\]

where \(G\) is the arc-node incidence matrix of the network.
Matching in Bipartite Graphs

**Theorem:** A graph is bipartite if and only if its node-edge incidence matrix is totally unimodular.


**König-Egervary Theorem:** Let $G$ be a bipartite graph. The maximum number of arcs in a matching is equal to the minimum number of nodes in a covering of arcs by nodes.

**Why?** By LP duality.
Reference


Reference

