Abstract. The logarithmic-exponential (log-exp) function has been widely used in convex
analysis and mathematical programming. This paper studies a natural generalization of the
log-exp function. Certain necessary and sufficient conditions are obtained for establishing such
a generalization. The derived sufficient conditions are explicitly expressed in terms of the
first and second derivatives of the functions involved, and hence can be easily checked. We
show that some classes of convex functions with certain regularity (such as $S^*$-regularity and
self-regularity) can be used to construct such generalized log-exp functions.

Key Words: Convexity, self-concordant functions, $S^*$-regular functions, self-regular func-
tions, differential inequality, entropy function, mathematical programming.

AMS Subject Classifications: 90C30, 90C25, 52A41, 49J52
1 Introduction

In this paper, we denote the $n$-dimensional Euclidean space by $R^n$, its nonnegative orthant by $R^n_+$, and positive orthant by $R^n_{++}$. The logarithmic-exponential (log-exp) function from $R^n$ to $R$ is given as

$$f(x) = \log \left( \sum_{i=1}^{n} e^{x_i} \right)$$

(1)

for $x = (x_1, ..., x_n)^T \in R^n$.

As shown in Rockafellar [20], the log-exp function is a convex function whose conjugate becomes the Shannon’s entropy function [21]:

$$g(x) = \begin{cases} 
\sum_{i=1}^{n} x_i \log x_i & \text{for } x_i \geq 0, i = 1, ..., n, \text{ and } \sum_{i=1}^{n} x_i = 1, \\
\infty, & \text{otherwise.}
\end{cases}$$

Shannon’s entropy function plays a vital role in so many fields ranging from the image enhancement to economics and from statistical mechanics to nuclear physics (see, for example, Buck and Macaulay [6] and Fang et al. [11]).

Being the conjugate of Shannon’s entropy function, the log-exp function itself has been widely used in convex analysis and mathematical programming. We name a couple of its important applications here.

First, let us consider a nonconvex “exponent function” mapping from $R^n$ to $R$ in the following form,

$$g(x) = \sum_{k=1}^{K} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \cdots x_n^{\alpha_{nk}}$$

(2)

where $c_k \geq 0$ and $\alpha_{ik} \in R$ for $k = 1, ..., K$ and $i = 1, ..., n$. A geometric program (see, for instance, Duffin et al.[7], Ecker [8], and Boyd and Vandenberghe [5]) is an optimization problem of the form

minimize $g_0(x)$

subject to $g_i(x) \leq 1, \quad i = 1, ..., l,$
$I_l+j(x) = 1, \quad j = 1, ..., m - l,$
$x_i > 0, \quad i = 1, ..., n$

where $g_0, ..., g_l$ are in the form of (2) and $g_{l+1}, ..., g_m$ are in the form of (2) with $K = 1$. By setting $y_i = \log x_i$ and taking the logarithm of the function (2), we can convert (2) into a convex function

$$d(y) = \log \left( \sum_{k=1}^{K} c_k e^{\alpha_{1k}^T y} \right) = \log \left( \sum_{k=1}^{K} e^{\alpha_{1k}^T y + b_k} \right)$$

where $a_k = (\alpha_{1k}, ..., \alpha_{nk})^T$ and $b_k = \log c_k$. Consequently, a geometric program can be converted into a convex programming problem using the log-exp function such that the interior-point algorithms can be developed to solve geometric programs with great efficiency (Nesterov...
and Nemirovsky [16] and Kortanek et al. [14]).

Second, let us consider a nondifferentiable minimax optimization problems in the following form
\[ \min_{y \in D} \max_{1 \leq i \leq n} g_i(y) \]
where \( g_i(\cdot), i = 1, ..., n \), is a real function defined on a convex set \( D \) in \( \mathbb{R}^m \). Notice that (Rockafellar [20]) the recession function of the log-exp function (1) is the “max-function”, i.e.,
\[ \max_{1 \leq i \leq n} x_i = \lim_{\varepsilon \to 0^+} \varepsilon f\left(\frac{x}{\varepsilon}\right) = \lim_{\varepsilon \to 0^+} \varepsilon \log \left( \sum_{i=1}^{n} e^{x_i/\varepsilon} \right). \]
Replacing \( x_i \) by \( g_i(y) \), we have
\[ \max_{1 \leq i \leq n} g_i(y) = \lim_{\varepsilon \to 0^+} \varepsilon \log \left( \sum_{i=1}^{n} e^{g_i(y)/\varepsilon} \right). \]
Consequently, a nondifferentiable minimax problem can be approximated by solving the following problem
\[ \min_{y \in D} \varepsilon \log \left( \sum_{i=1}^{n} e^{g_i(y)/\varepsilon} \right), \]
which is differentiable and convex if every \( g_i(y) \) is. This smoothing technique was frequently used (see, for example, Ben-Tal and Teboulle [1], Bersekas [2], Li and Fang [15], Polyak [19], and Zang [25]). It is worth mentioning that the error bound analysis and the relationship between the primal approximate problem using the log-exp function and its corresponding dual were included in Ben-Tal and Teboulle [1]. Similar investigation has been conducted by several other authors in a variety of circumstances. For example, it was used by Fang [9, 10] to study linear programming problems; by Birbil et al. [3] to derive a recursive approximation of the higher dimensional max-function; by Peng and Lin ([17]) and Fang et al. [12] to study vertical complementarity problems; by Sun and Li [24, 22, 23] to design value-estimation function for bounded global optimization problems, and by Sun and Li [23, 24] to study the dual formulation of integer programming problems.

In view of the importance of the log-exp function (1), we aim to generalize it and construct a new class of the generalized log-exp functions. Such a generalization may not only yield new theoretical results in convex analysis, but also allow us to tackle different optimization problems by choosing the most efficient one from among generalized log-exp-like functions.

Observe that the log-exp function (1) is a convex function formed by the exponential function \( e^t \), which is a convex, strictly increasing function defined on \( R \), and its inverse function \( \log t \), which is a concave, increasing function defined on \( R_{++} \). A simple generalization leads us to consider the following function
\[ \Upsilon_w(x) = \phi^{-1} \left( \sum_{i=1}^{n} w_i \phi(x_i) \right) \]
where \( \phi \) is a real, convex, strictly increasing function defined on a subset of \( \mathbb{R}^n \) and \( w = [w_1, w_2, \ldots, w_n]^T \) is a given vector in \( \mathbb{R}^n_+ \). Note that the log-exp and \( p \)-norm functions fit right in this category. A fundamental question is “What kind of \( \phi \) functions can make \( \Upsilon_w \) convex?” Our investigation will lead to an interesting characterization of these \( \phi \) functions.

A further generalization is to consider the following function:

\[
\Gamma_w(x) = \Psi^{-1} \left( \sum_{i=1}^{n} w_i \phi_i(x_i) \right)
\]

where \( \phi_i : \Omega \to R, i = 1, \ldots, n, \) is a convex, twice differentiable (but not necessarily being strictly increasing) function defined on an open convex set \( \Omega \subset R \), \( \Psi : \Omega \to R \) is convex, twice differentiable and strictly increasing, and \( w \in \mathbb{R}^n_+ \) is a given vector. Clearly, \( \Upsilon_w(x) \) is a special case of \( \Gamma_w(x) \) with \( \phi_1 = \phi_2 = \ldots = \phi_n = \Psi = \phi \). Our second fundamental question is “What are the sufficient and necessary conditions for the generalized log-exp functions \( \Gamma_w \) being convex?” Moreover, if possible, we would like to find a systematic way to construct explicitly such new classes of generalized log-exp functions \( \Gamma_w(x) \).

For simplicity and convenience, in this paper, we call \( \phi_i(t) \) the inner function and \( \Psi \) the outer function of \( \Gamma_w(x) \). To assure the well-definedness of \( \Gamma_w(x) \), we assume that

\[
\sum_{i=1}^{n} \text{Cone}[\phi_i(\Omega)] \subseteq \Psi(\Omega)
\]

where \( \text{Cone}[\phi_i(\Omega)] \) denotes the cone generated by the set \( \phi_i(\Omega) \).

It is interesting to point out that our second fundamental question is closely related to a more general problem raised by W. Fenchel in his lecture notes of “Convex cones, sets and functions” in 1953 [13]. Fenchel’s question was “For a given family of levels sets, under what conditions, can they be transformable into the family of level sets of a convex function?” Based on the properties of level sets and characteristic roots of Hessian matrices involved, Fenchel derived some sufficient and necessary conditions for smooth convex functions with prescribed level sets. The conditions he derived are rather complicated, and there is no simple test to decide what kind of functions may admit these complicated properties. Therefore, Fenchel’s result is not directly applicable for answering our questions and systematically constructing generalized log-exp functions. Unlike Fenchel’s approach, our analysis in this paper depends only on the function value, its first derivative, and second derivative to provide an answer to the above-mentioned fundamental questions. The necessary and sufficient conditions we derived can be easily tested, thus making it possible for us to explicitly construct new classes of generalized log-exp functions. In particular, we show that how the self-regular functions [18] (most of such functions are self-concordant functions as defined in [16]), and \( S^* \)-regular functions (to be defined in this paper) can be used to construct generalized log-exp functions.
The rest of the paper is organized as follows. In Section 2, we investigate the conditions
that assure the convexity of the generalized log-exp functions. The results answer our second
fundamental question. In Section 3, we provide an answer to our first fundamental question
and identify some classes of functions that satisfy the conditions derived in Section 2. In
Section 4, we illustrate how the generalized log-exp functions can be explicitly constructed.
Some final remarks and future work are given in the last section.

2 Necessary and Sufficient Conditions for Convexity of $\Gamma_w(x)$

Let us start with a simple lemma that shows the inverse function of an increasing convex
function is concave and increasing.

**Lemma 2.1.** Let $\Omega$ be an open convex subset of $R$ and $\Psi : \Omega \rightarrow R$ be a real function
defined on $\Omega$. Then $\Psi$ is convex and strictly increasing if and only if its inverse $\Psi^{-1} : R \rightarrow \Omega$
is concave and strictly increasing.

**Proof.** When $\Psi : \Omega \rightarrow R$ is strictly increasing, it is easy to see that $\Psi^{-1} : R \rightarrow \Omega$ is also
strictly increasing. For $\Psi$ being convex, we have

$$\Psi(\lambda \hat{t} + (1 - \lambda)\check{t}) \leq \lambda \Psi(\hat{t}) + (1 - \lambda)\Psi(\check{t}) \quad \text{for any } \hat{t}, \check{t} \in \Omega \text{ and } \lambda \in [0, 1].$$

Let $\tilde{y} = \Psi(\hat{t})$ and $\check{y} = \Psi(\check{t})$. For any $\lambda \in [0, 1]$, since $\Psi^{-1}$ is increasing,

$$\Psi^{-1}(\lambda \tilde{y} + (1 - \lambda)\check{y}) = \Psi^{-1}(\lambda \Psi(\hat{t}) + (1 - \lambda)\Psi(\check{t})) \geq \Psi^{-1}(\Psi(\lambda \hat{t} + (1 - \lambda)\check{t})) = \lambda \tilde{y} + (1 - \lambda)\check{y} = \lambda \Psi^{-1}(\tilde{y}) + (1 - \lambda)\Psi^{-1}(\check{y}).$$

Hence $\Psi^{-1}$ is concave. The converse statement can be shown in a similar manner. $\square$

To study the convexity of the generalized log-exp function $\Gamma_w(x)$, when assuming that $\phi_i$, $i = 1, ..., n$, and $\Psi^{-1}$ are twice differentiable, we need to check its Hessian matrix. Let

$$x_w = \sum_{i=1}^{n} w_i \phi_i(x_i).$$

Since $\frac{\partial x_w}{\partial x_i} = w_i \phi'_i(x_i)$, we have

$$\frac{\partial \Gamma_w}{\partial x_i} = (\Psi^{-1})'(x_w)w_i \phi'_i(x_i).$$

Moreover,

$$\frac{\partial^2 \Gamma_w}{\partial x_i^2} = (\Psi^{-1})''(x_w)(w_i \phi'_i(x_i))^2 + (\Psi^{-1})'(x_w)w_i \phi''_i(x_i),$$
\[
\frac{\partial^2 \Gamma_w}{\partial x_i \partial x_j} = (\Psi^{-1})''(x_w) w_i w_j \phi'_i(x_i) \phi'_j(x_j) \quad \text{for } i \neq j.
\]

Consequently, the Hessian matrices of \( \Gamma_w \) becomes
\[
\frac{\partial^2 \Gamma_w}{\partial x^2} = (\Psi^{-1})'(x_w) \begin{bmatrix}
    w_1 \phi''_1(x_1) & 0 & \cdots & 0 \\
    0 & w_2 \phi''_2(x_2) & \cdots & 0 \\
    \cdots & \cdots & \cdots & \cdots \\
    0 & 0 & \cdots & w_n \phi''_n(x_n)
\end{bmatrix} + (\Psi^{-1})''(x_w) \begin{bmatrix}
    w_1 \phi'_1(x_1) \\
    w_2 \phi'_2(x_2) \\
    \cdots \\
    w_n \phi'_n(x_n)
\end{bmatrix} \begin{bmatrix}
    w_1 \phi'_1(x_1), w_2 \phi'_2(x_2), \ldots, w_n \phi'_n(x_n)
\end{bmatrix}.
\]

Note that when \( \phi_i, i = 1, \ldots, n, \) is convex and \( \Psi \) is convex and increasing, by Lemma 2.1, we see that the first term on the right-hand side of (5) is a positive semidefinite matrix multiplied by a positive coefficient \((\Psi^{-1})'(x_w)\), while the second is of rank one and is multiplied by a negative coefficient \((\Psi^{-1})''(x_w)\).

Notice that if \( w_i = 0 \) for some \( i \), then the term \( w_i \phi_i(x) \) can be removed from the function \( \Gamma_w(x) \). For this case, it suffices to consider \( \Gamma_w(x) \) defined on \( R_{++}^n \). Thus, without loss of generality, we may assume that the vector \( w \in R_{++}^n \) throughout the rest of this paper. Using the first and second order information of \( \phi_i \) and \( \Psi \), we have the following key result:

**Theorem 2.1.** Let \( \Omega \subset R \) be open and convex, \( \Psi : \Omega \rightarrow R \) be convex, twice differentiable and strictly increasing, \( \phi_i : \Omega \rightarrow R, \ i = 1, \ldots, n, \) be strictly convex and twice differentiable, and \( w \in R_{++}^n \) be a given vector. Then the generalized log-exp function
\[
\Gamma_w(x) = \Psi^{-1} \left( \sum_{i=1}^n w_i \phi_i(x) \right)
\]
is convex on \( \Omega^n := \underbrace{\Omega \times \cdots \times \Omega}_{n} \) if and only if
\[
\Psi''(y) \left( \sum_{i=1}^n w_i \left( \frac{\phi'_i(x_i)}{\phi''_i(x_i)} \right)^2 \right) \leq [\Psi'(y)]^2 \quad \text{for } x \in \Omega^n \text{ and } y = \Gamma_w(x).
\]

**Proof.** Let \( y = \Gamma_w(x) = \Psi^{-1}(x_w) \). Then \( x_w = \Psi(y) \) and
\[
(\Psi^{-1})'(x_w) \Psi'(y) = 1.
\]
Differentiating both sides of (7) with respect to \( y \) and use the above relations, we have
\[
0 = (\Psi^{-1})''(x_w) [\Psi'(y)]^2 + (\Psi^{-1})'(x_w) \Psi''(y) = (\Psi^{-1})''(x_w) [\Psi'(y)]^2 + \frac{\Psi''(y)}{\Psi'(y)}.
\]
Therefore,

\[(\Psi^{-1})''(x_w) = -\frac{\Psi''(y)}{|\Psi'(y)|^3}.\]  

(8)

Combining (7) and (8) yields

\[(\Psi^{-1})'(x_w) + (\Psi^{-1})''(x_w) \sum_{i=1}^{n} w_i [\phi_i'(x_i)]^2 = \frac{[\Psi'(y)]^2 - \left( \sum_{i=1}^{n} w_i \frac{[\phi_i'(x_i)]^2}{\phi_i''(x_i)} \right) \Psi''(y)}{|\Psi'(y)|^3}.\]  

(9)

First we prove that \(\Gamma_w(x)\) is convex, if (6) holds. It suffices to show that the Hessian matrix of \(\Gamma_w(x)\) is positive semi-definite.

Notice that for any \(d \in \mathbb{R}^n\) and \(x \in \Omega^n\), the Cauchy-Schwartz inequality implies that

\[\left( \sum_{i=1}^{n} w_i \phi_i'(x_i) d_i \right)^2 \leq \left( \sum_{i=1}^{n} w_i \phi_i''(x_i) d_i \right) \left( \sum_{i=1}^{n} w_i \frac{[\phi_i'(x_i)]^2}{\phi_i''(x_i)} \right).

By Lemma 2.1, we know \(\Psi^{-1}\) is concave and hence \((\Psi^{-1})''(x_w) \leq 0\). Combining this fact with the above inequality, we see that, for any \(d \in \mathbb{R}^n\),

\[d^T \frac{\partial^2 \Gamma_w}{\partial x^2} d \]

\[= (\Psi^{-1})'(x_w) \left( \sum_{i=1}^{n} w_i \phi_i''(x_i) d_i^2 \right) + (\Psi^{-1})''(x_w) \left( \sum_{i=1}^{n} w_i \phi_i'(x_i) d_i \right)^2 \]

\[\geq (\Psi^{-1})'(x_w) \left( \sum_{i=1}^{n} w_i \phi_i''(x_i) d_i^2 \right) + (\Psi^{-1})''(x_w) \left( \sum_{i=1}^{n} w_i \frac{[\phi_i'(x_i)]^2}{\phi_i''(x_i)} \right) \left( \sum_{i=1}^{n} w_i \phi_i'(x_i) d_i \right)^2 \]

\[= \left( \sum_{i=1}^{n} w_i \phi_i'(x_i) d_i \right) \left[ (\Psi^{-1})'(x_w) + (\Psi^{-1})''(x_w) \left( \sum_{i=1}^{n} w_i \frac{[\phi_i'(x_i)]^2}{\phi_i''(x_i)} \right) \right] \]

\[\geq 0.

The last equality follows from (9) and the last inequality follows from the fact that the first quantity on the right-hand side, i.e., \(\sum_{i=1}^{n} w_i \phi_i''(x_i) d_i^2\), is nonnegative, and the second quantity is also nonnegative due to our assumption. Consequently, we have proven that the Hessian matrix \(\frac{\partial^2 \Gamma_w}{\partial x^2}\) is positive semi-definite, as desired.

Conversely, we would like to show inequality (6) holds, if \(\Gamma_w(x)\) is convex. For any vector \(0 \neq d \in \mathbb{R}^n\), knowing (7), (8) and the convexity of \(\Gamma_w(x)\), we have

\[0 \leq d^T \frac{\partial^2 \Gamma_w}{\partial x^2} d = (\Psi^{-1})'(x_w) \left( \sum_{i=1}^{n} w_i \phi_i''(x_i) d_i^2 \right) + (\Psi^{-1})''(x_w) \left( \sum_{i=1}^{n} w_i \phi_i'(x_i) d_i \right)^2.

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the stronger the convexity. (For example, this index function in a sense indicates the degree of convexity, namely, the smaller the index, the easier it is to construct generalized log-exp functions. First, let us introduce an interesting index function $\Psi_0(y) := \frac{1}{\Psi'(y)} \left( \sum_{i=1}^{n} w_i \phi_i''(x_i) d_i^2 \right) - \frac{\Psi''(y)}{\Psi'(y)^3} \left( \sum_{i=1}^{n} w_i \phi_i'(x_i) d_i \right)^2$

$$= \left( \sum_{i=1}^{n} w_i \phi_i''(x_i) d_i^2 \right) \left[ \frac{1}{\Psi'(y)} \right] - \frac{\Psi''(y)}{\Psi'(y)^3} \left[ \sum_{i=1}^{n} w_i \phi_i'(x_i) d_i \right]^2]. \quad (10)$$

Let $U$ and $Z$ be two vectors in $R^n$ defined by

$$U := \left( \sqrt{w_1 \phi_1''(x_1)} d_1, \sqrt{w_2 \phi_2''(x_2)} d_2, \ldots, \sqrt{w_n \phi_n''(x_n)} d_n \right)^T$$

and

$$Z := \left( \sqrt{w_1 \phi_1'(x_1)} \phi_1'(x_1), \sqrt{w_2 \phi_2'(x_2)} \phi_2'(x_2), \ldots, \sqrt{w_n \phi_n'(x_n)} \phi_n'(x_n) \right)^T.$$

Since $w \in R^n_{++}$ and $\phi$ is strictly convex, inequality (10) reduces to

$$\langle U^T U \rangle \left[ \frac{\Psi'(y)}{\Psi'(y)^3} \right] (U^T Z)^2 \geq 0 \text{ for any } 0 \neq U \in R^n,$$

i.e.,

$$[\Psi'(y)]^2 \geq \frac{(U^T Z)^2}{U^T U} \Psi''(y) \text{ for all } 0 \neq U \in R^n.$$

Therefore, we have

$$[\Psi'(y)]^2 \geq \sup_{0 \neq U \in R^n} \frac{(U^T Z)^2}{U^T U} \Psi''(y) = \sup_{V^T V = 1} (V^T Z)^2 \Psi''(y) = [(V^*)^T Z]^2 \Psi''(y),$$

where $V^* = Z/\|Z\|_2$ is the optimal solution of the problem $\sup_{V^T V = 1} (V^T Z)^2$. Hence we have

$$[\Psi'(y)]^2 \geq [(V^*)^T Z]^2 \Psi''(y) = Z^T Z \Psi''(y).$$

Using the definition of $Z$, we see that inequality (6) indeed holds. This completes the proof. $\square$

The above theorem answers our second fundamental question. We now study how this result can help us construct generalized log-exp functions. First, let us introduce an interesting index function $\alpha(t)$ for a twice differentiable convex function $\delta(t)$ defined on a convex subset $\Omega$ of $R$:

$$\alpha(t) = \frac{[\delta'(t)]^2}{\delta(t) \delta''(t)} \text{ for } t \in \Omega.$$ 

This index function in a sense indicates the degree of convexity, namely, the smaller the index, the stronger the convexity. (For example, $\alpha(t) = 2$ for $\delta(t) = t^2$ on $[0, \infty)$, and $\alpha(t) = 1$ for $\delta(t) = e^t$ on $R$. ) Similar arguments can be made for concave functions. In this paper, we particularly consider those convex functions having bounds on the index function, i.e., $\alpha(t) \geq \alpha$ or $\alpha(t) \leq \alpha$ over $\Omega$, for some $\alpha \in R$. The bounds of the index function will lead us to the construction of generalized log-exp functions through the following two results:
Theorem 2.2. Let $\Omega$ be an open convex subset of $\mathbb{R}$, $\Psi : \Omega \to \mathbb{R}$ be strictly increasing, twice differentiable and convex, $\phi_i : \Omega \to \mathbb{R}$, $i = 1, ..., n$, be strictly convex and twice differentiable, and $w \in \mathbb{R}_+^n$ be a given vector. Assume that there exists a scalar $\alpha \in \mathbb{R}$ such that

$$\alpha \Psi(t)\Psi''(t) \leq [\Psi'(t)]^2$$

for $t \in \Omega$. (11)

Then the generalized log-exp function $\Gamma_w(x)$ is convex on $\Omega^n$ if

$$\sum_{i=1}^{n} \frac{w_i[\phi_i'(x_i)]^2}{\phi_i''(x_i)} \leq \alpha \Psi(y)$$

for $x \in \Omega^n$, (12)

where $y = \Gamma_w(x)$.

Proof. Multiplying both sides of (12) by $\Psi''(y)$ and applying (11), we see that condition (6) holds. Then the result follows from Theorem 2.1 immediately. □

Theorem 2.3. Let $\Omega$ be an open convex subset of $\mathbb{R}$, $\Psi : \Omega \to \mathbb{R}$ be strictly increasing, twice differentiable and convex, $\phi_i : \Omega \to \mathbb{R}$, $i = 1, ..., n$, be strictly convex and twice differentiable, and $w \in \mathbb{R}_+^n$ be a given vector. Assume that there exist $0 \neq \alpha_i \in \mathbb{R}$, $i = 1, ..., n$, holding the same sign such that

$$\alpha_i \phi_i(t)\phi''_i(t) \geq [\phi_i'(t)]^2$$

for $t \in \Omega$, (13)

and there exists an $\alpha \in \mathbb{R}$ such that the inequality (11) holds. Then the generalized log-exp function $\Gamma_w(x)$ is convex if

$$\alpha \geq \max_{1 \leq i \leq n} \alpha_i \quad \text{(when } \alpha_i > 0 \text{ for all } i), \quad (14)$$

or

$$\alpha \leq \min_{1 \leq i \leq n} \alpha_i \quad \text{(when } \alpha_i < 0 \text{ for all } i). \quad (15)$$

Proof. Taking $y = \Gamma_w(x)$, We see two cases.

Case 1: $\alpha_i > 0$ for $i = 1, ..., n$. In this case, (13) implies that $\phi_i(t) \geq 0$ for $t \in \Omega$ and (14) implies that

$$\sum_{i=1}^{n} w_i \frac{[\phi_i'(t)]^2}{\phi_i''(x_i)} \leq \sum_{i=1}^{n} w_i \alpha_i \phi_i(x_i) \leq \left( \max_{1 \leq i \leq n} \alpha_i \right) \sum_{i=1}^{n} w_i \phi_i(x_i) \leq \alpha \Psi(y).$$

Case 2: $\alpha_i < 0$ for $i = 1, ..., n$. In this case, (13) implies that $\phi_i(t) \leq 0$ for $t \in \Omega$ and (15) implies that

$$\sum_{i=1}^{n} w_i \frac{[\phi_i'(t)]^2}{\phi_i''(x_i)} \leq \sum_{i=1}^{n} w_i \alpha_i \phi_i(x_i) \leq \left( \min_{1 \leq i \leq n} \alpha_i \right) \sum_{i=1}^{n} w_i \phi_i(x_i) \leq \alpha \Psi(y).$$

Both cases yield condition (12) and the desired result follows from Theorem 2.2. □
One special case is that $\phi_1(t) = \phi_2(t) = \ldots = \Psi(t) = \phi(t)$ and both inequalities (11) and (13) hold with an equality. In this case, $\alpha_i = \alpha$ for $i = 1, \ldots, n$. (For example, when $\phi_1(t) = \phi_2(t) = \ldots = \Psi(t) = \phi(t) = e^t$, we have $\alpha_i = 1$ for $i = 1, \ldots, n$ and $\Psi^{-1}(t) = \log t$.) This leads to the next result.

**Corollary 2.1.** Let $\Omega$ be an open convex set in $R$, $\phi : \Omega \to R$ be a convex, twice differentiable and strictly increasing function, and $w \in R^n_{++}$ be a given vector. If there exists an $\alpha \neq 0$ such that
\[ |\phi'(t)|^2 = \alpha \phi(t) \phi''(t) \quad \text{for} \quad t \in \Omega, \tag{16} \]
then the generalized log-exp function $\Upsilon_w(x) = \phi^{-1}(\sum_{i=1}^{n} w_i \phi(x_i))$ is convex on $\Omega^n$.

**Proof.** By Theorem 2.3, it suffices to prove that $\phi$ is strictly convex. Since $\phi$ is strictly increasing, $\phi'(t) > 0$ on $\Omega$. From (16), we see that $\phi(t) \neq 0$ and $\phi''(t) \neq 0$ for $t \in \Omega$. By convexity, $\phi''(t)$ must be positive on $\Omega$. Hence $\phi$ is strictly convex and the result follows immediately from Theorem 2.3. $\Box$

It is worthy of mentioning two remarks here:

**Remark 2.1.** The functions satisfying a differential inequality like (11) are related to the so-called self-concordant barrier function as introduced in Nesterov and Nemirovsky [16]. Recalled that a $C^3$ function $\xi : (0, \infty) \to R$ is said to be self-concordant if $\xi$ is convex and there exists a constant $\mu_1 > 0$ such that
\[ |\xi'''(t)| \leq \mu_1 (\xi''(t))^\frac{3}{2} \quad \text{for} \quad t \in (0, \infty). \tag{17} \]
Moreover, the self-concordant function $\xi$ is called a self-concordant barrier function if there exists a constant $\mu_2 > 0$ such that
\[ |\xi'(t)| \leq \mu_2 [f''(t)]^\frac{1}{2} \quad \text{for} \quad t \in (0, \infty). \tag{18} \]
Combination of (17) and (18) yields
\[ \xi'(t)\xi'''(t) \leq \mu [\xi''(t)]^2. \]
This indicates that the first-order derivative function of a self-concordant barrier function, i.e., $g(t) := \xi'(t)$, satisfies the inequality (11). Our later analysis will point out that a self-concordant function $\xi(t)$ itself may also satisfy an inequality like (11) or (13).

**Remark 2.2.** The functions satisfying a differential inequality like (11) also appear in convexity theory. Given a twice differentiable function $\phi(t) > 0$ on its domain $\Omega$, we consider the convexity of the function $h(t) := \frac{1}{\phi(t)}$ on $\Omega$. Notice that
\[ h''(t) = \frac{2[\phi'(t)]^2 - \phi(t)\phi''(t)}{[\phi(t)]^3} \quad \text{for} \quad t \in \Omega. \]
Hence the function \( h(t) = \frac{1}{\phi(t)} \) is convex if and only if the inequality \( \phi(t)\phi''(t) \leq 2[\phi'(t)]^2 \) holds on \( \Omega \). Moreover, the convex function \( h(t) \) satisfies a reverse inequality, i.e., \( h(t)h''(t) \geq [h'(t)]^2 \) on \( \Omega \). However, note that the condition \( \phi(t)\phi''(t) \leq 2[\phi'(t)]^2 \) is no longer sufficient for the convexity of \( \frac{1}{\sqrt{\phi(t)}} \). Actually, it requires that \( \phi(t)\phi''(t) \leq \frac{3}{2}[\phi'(t)]^2 \) for the convexity of \( \frac{1}{\sqrt{\phi(t)}} \).

From this observation, a related question arises: Given a function \( \phi(t) > 0 \) on \( \Omega \) and a constant \( r > 0 \), when will the function \( h(t) := \frac{1}{\phi(t)^r} \) become convex and satisfy an inequality like (13)? A straightforward analysis leads to the next result.

**Theorem 2.4.**

(A) Let \( \Omega \) be a convex subset of \( R \) and \( \phi : \Omega \to (0, \infty) \) be a function. If \( \phi(t)\phi''(t) \leq [\phi'(t)]^2 \) for \( t \in \Omega \), then, for any \( r > 0 \), the function \( h(t) := \frac{1}{\phi(t)^r} \) is convex with \( h(t)h''(t) \geq [h'(t)]^2 \) for \( t \in \Omega \). Conversely, if there exists an \( r > 0 \) such that \( h(t) := \frac{1}{\phi(t)^r} \) is convex with \( h(t)h''(t) \geq [h'(t)]^2 \) for \( t \in \Omega \), then \( \phi(t)\phi''(t) \leq [\phi'(t)]^2 \) for \( t \in \Omega \).

(B) Let \( \Omega \) be a convex subset of \( R \), \( \tau > 0 \), and \( \phi : \Omega \to (\tau, \infty) \) be a function. If \( \phi(t)\phi''(t) \leq [\phi'(t)]^2 \) for \( t \in \Omega \), then, for any \( r > 0 \) and \( T > 0 \), the function \( h_T(t) := T + \frac{1}{\phi(t)^r} \) is convex with \( \alpha h_T(t)h''_T(t) \geq [h'_T(t)]^2 \) for \( t \in \Omega \), where \( \alpha = \frac{1}{T\tau^r+1} \).

**Proof.** For case (A), it is sufficient to see that

\[
h''(t) = \frac{r^2(\phi'(t))^2 + r[(\phi'(t))^2 - \phi(t)\phi''(t)]}{\phi(t)^{r+2}},
\]

and

\[
h(t)h''(t) - [h'(t)]^2 = \frac{r[(\phi'(t))^2 - \phi(t)\phi''(t)]}{\phi(t)^{2(r+1)}}.
\]

For case (B), it is easy to verify that \( h''_T(t) = h''(t) \) and

\[
\left( \frac{1}{T\phi(t)^r} + 1 \right) h_T(t)h''_T(t) - [h'_T(t)]^2 = \frac{r[(\phi'(t))^2 - \phi(t)\phi''(t)]}{\phi(t)^{2(r+1)}}.
\]

Then the desired result follows. \( \square \)

The above results indicate that if we have a function \( \phi \) satisfying the inequality (11) with \( \alpha = 1 \), then we may construct a function \( h \) from \( \phi \) such that \( h \) satisfies the converse differentiable inequality \( \alpha h(t)h''(t) \geq [h'(t)]^2 \) for some constant \( \alpha \). Moreover, if we take a \( T \)-translation of the value of the function \( h \), then the resulting function satisfies the converse differentiable inequality with an \( \alpha \) that can be reduced to be smaller than any threshold given in (0,1) provided a suitable \( T > 0 \) is chosen. This fact will be used in Section 4.

### 3 Functions of interest

In this section, we try to identify some classes of functions that satisfy inequality (11) and/or inequality (13). First, let us identify functions that satisfy the equation (16). Obviously, this class of functions satisfy both inequalities (11) and (13). 

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Theorem 3.1. Let $\Omega$ be an open set in $\mathbb{R}$ and $\phi : \Omega \to \mathbb{R}$ be a convex, twice differentiable and strictly increasing function satisfying equation (16) with a constant $\alpha \neq 0$. Then,

(i) when $\alpha = 1$, $\phi(t)$ is in the form of

$$\phi(t) = \gamma e^{\frac{t}{\beta}}$$

for some $\gamma > 0$ and $\beta > 0$.

(ii) when $0 < \alpha \neq 1$, $\phi(t)$ is in the form of

$$\phi(t) = \gamma \left( \frac{1}{\alpha} - \frac{1}{\alpha} t + \beta \right)^{\frac{\alpha}{\alpha-1}}$$

for some $\gamma > 0$ and $\beta \geq v^*$.

(iii) when $\alpha < 0$, $\phi(t)$ is in the form of

$$\phi(t) = -\gamma \left( \beta - \frac{1}{\alpha} - \frac{1}{\alpha} t \right)^{\frac{\alpha}{\alpha-1}}$$

for some $\gamma > 0$ and $\beta \geq u^*$.

Proof. As we have pointed out in the proof of Corollary 2.1, equality (16) implies that $\phi(t) \neq 0$ and $\phi''(t) > 0$ on $\Omega$. Actually, from (16), we see that $\phi(t) > 0$ if $\alpha > 0$, and $\phi(t) < 0$ if $\alpha < 0$. Notice that

$$\left[ \frac{\phi(t)}{\phi'(t)} \right]' = 1 - \frac{\phi(t)\phi''(t)}{[\phi'(t)]^2} = 1 - \frac{1}{\alpha},$$

Consequently, we have

$$\frac{\phi(t)}{\phi'(t)} = (1 - \frac{1}{\alpha})t + \beta \quad \text{for} \quad t \in \Omega,$$

where $\beta$ is a constant.

(i) If $\alpha = 1$, we know $\phi(t) > 0$, $\phi'(t) > 0$, and (19) reduces to $\phi'(t)/\phi(t) = 1/\beta$ with some $\beta > 0$. Hence,

$$\phi(t) = \gamma e^{\frac{t}{\beta}}$$

for some $\gamma > 0$.

(ii) If $0 < \alpha \neq 1$, we know $\phi(t) > 0$. It follows from (19) that $\beta$ should be taken such that

$$(1 - \frac{1}{\alpha})t + \beta > 0 \quad \text{for} \quad t \in \Omega.$$ When $v^* := \sup_{t \in \Omega} \frac{1-\alpha}{\alpha} t$ is finite, it suffices to take any $\beta \geq v^*$. In this case, it follows from (19) that

$$\int \frac{\phi'(t)}{\phi(t)} dt = \int \frac{1}{(1 - \frac{1}{\alpha})t + \beta} dt.$$

Consequently, we have

$$\phi(t) = \gamma \left( \frac{1}{\alpha} - \frac{1}{\alpha} t + \beta \right)^{\frac{\alpha}{\alpha-1}}$$

for some $\gamma > 0$. 

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(iii) If \( \alpha < 0 \), we know \( \phi(t) < 0 \). From (19), \( \beta \) should be chosen such that \((1 - \frac{1}{\alpha})t + \beta < 0 \) for \( t \in \Omega \). When \( u^* := \sup_{t \in \Omega} \frac{\alpha - 1}{\alpha} t \) is finite, it suffice to take any \( \beta \leq -u^* \). From (19), we have

\[
\int \frac{-\phi''(t)}{-\phi(t)} dt = - \int \frac{1}{-(1 - \frac{1}{\alpha})t + \beta} dt.
\]

Thus, \( \ln(-\phi(t)) = \frac{\alpha}{\alpha - 1} \ln[-(1 - \frac{1}{\alpha})t - \beta] + c \) for some constant \( c \). Consequently, we have

\[
\phi(t) = -\gamma \left(-\frac{\alpha - 1}{\alpha} t - \beta \right)^{\frac{\alpha}{\alpha - 1}},
\]

for some \( \gamma > 0 \). Replacing \(-\beta\) by \( \beta \) leads to result (iii). \( \Box \)

Notice that for an open convex set \( \Omega \subset \mathbb{R} \),
(i) if \( \Omega = (\eta_1, \eta_2) \) with \( \eta_1 \) and \( \eta_2 \) being finite, then \( v^* \) and \( u^* \) in Theorem 3.1 are finite.
(ii) if \( \Omega = (\eta, \infty) \) with \( \eta \) being finite, then \( v^* = (1 - \alpha)\eta/\alpha \) for any \( \alpha > 1 \).
(iii) if \( \Omega = (-\infty, \eta) \) with \( \eta \) being finite, then \( v^* = (1 - \alpha)\eta/\alpha \) for any \( \alpha \in (0, 1) \).
(iv) if \( \Omega = (-\infty, \eta) \) with \( \eta \) being finite, then \( u^* = (1 - \alpha)\eta/\alpha \) for any \( \alpha < 0 \).

Subsequently, we can easily see the following result which actually answers our first fundamental question.

**Corollary 3.1.** The following functions can be used to explicitly construct a convex generalized log-exp function \( \Upsilon_w(x) = \phi^{-1}(\sum_{i=1}^{n} w_i \phi(x_i)) \) over \( \Omega^n \):

(i) \( \phi(t) = \gamma e^{\beta t} \) over \( \Omega = R \) with \( \gamma > 0 \) and \( \beta > 0 \).
(ii) \( \phi(t) = \gamma \left(\frac{1}{p} t + \beta\right)^p \) over \( \Omega = (\eta, \infty) \) with \( p > 1 \), \( \gamma > 0 \) and \( \beta \geq -\frac{\eta}{p} \).
(iii) \( \phi(t) = \gamma \left(\frac{t}{\beta - \frac{1}{p}}\right)^p \) over \( \Omega = (-\infty, \eta) \) with \( p > 0 \), \( \gamma > 0 \) and \( \beta \geq -\frac{\eta}{p} \).
(iv) \( \phi(t) = -\gamma(\beta - \frac{1}{p})^p \) over \( \Omega = (-\infty, \eta) \) with \( 0 < p < 1 \), \( \gamma > 0 \) and \( \beta \geq \frac{\eta}{p} \).

Notice that for case (i) with \( \gamma = \beta = 1 \), we have the regular log-exp function with \( \phi(t) = e^{t} \).
For case (ii) with \( p = 2 \), the quadratic function \( \phi(t) = \gamma(\frac{1}{p} t^2 + t \beta + \beta^2) \) becomes applicable.
Also for case (ii) with \( \eta = \beta = 0 \), \( \phi(t) = ct^p \) for some \( c > 0 \). Taking \( c = 1 \), then we have \( \Upsilon_w(x) = \|x\|_p \), i.e., the \( p \)-norm function. The functions listed in Corollary 3.1 form a complete basis in the sense that the function \( \phi \) in case (i) satisfies condition (16) with \( \alpha = 1 \); the function \( \phi \) in case (ii) satisfies condition (16) with \( \alpha = \frac{p}{p - 1} > 1 \); the function \( \phi \) in case (iii) satisfies condition (16) with \( \alpha = \frac{p}{p + 1} \in (0, 1) \); and the function \( \phi \) in case (iv) satisfies condition (16) with \( \alpha = \frac{p}{p + 1} < 0 \).

We now try to identify some class of functions that satisfy inequalities (11) and/or (13). For simplicity, we only consider convex, twice differentiable, strictly increasing functions \( \vartheta \) on \( \Omega = (0, \infty) \). Let us first define the following four categories of such functions:

\[ U_1 = \{ \vartheta : \text{There exists } \alpha \in \mathbb{R} \text{ such that } \alpha \vartheta(t) \vartheta''(t) \geq [\vartheta'(t)]^2 \text{ for } t \in \Omega \}. \]
\[ \mathcal{U}_2 = \{ \vartheta : \text{There exists } \alpha \in R \text{ such that } \alpha \vartheta(t) \vartheta''(t) \leq [\vartheta'(t)]^2 \text{ for } t \in \Omega \}; \]

\[ \mathcal{U}_3 = \{ \vartheta : \text{There exist } \alpha_1 \leq \alpha_2 \text{ such that } \alpha_1 \vartheta(t) \vartheta''(t) \leq [\vartheta'(t)]^2 \leq \alpha_2 \vartheta(t) \vartheta''(t) \text{ for } t \in \Omega \}; \]

\[ \mathcal{U}_4 = \{ \vartheta : \text{There exists } \alpha \in R \text{ such that } \alpha \vartheta(t) \vartheta''(t) = [\vartheta'(t)]^2 \text{ for all } t \in \Omega \}. \]

It is evident that

\[ \mathcal{U}_4 \subset \mathcal{U}_3 \subset (\mathcal{U}_2 \cap \mathcal{U}_1). \]

As pointed out in Theorem 3.1, the class \( \mathcal{U}_4 \) can be given explicitly. By allowing \( \alpha_1 \neq \alpha_2 \), we show that \( \mathcal{U}_3 \) is much broader than \( \mathcal{U}_4 \). In fact, many convex functions with certain regularities fall into the category \( \mathcal{U}_3 \).

To start, we introduce a new class of functions with certain regularity properties.

**Definition 3.1.** A convex, twice differentiable, strictly increasing function \( \delta(t) : (0, \infty) \rightarrow R \) is called an \( S^\ast \)-regular function if (i) \( \delta(t) \) vanishes at \( t = 0 \) in the sense of

\[ \lim_{t \to 0^+} \delta(0) = \lim_{t \to 0^+} \delta'(0) = \lim_{t \to 0^+} \delta''(0) = 0; \]

and (ii) there exist positive constants \( 0 < \beta_1 \leq \beta_2, p \geq 1 \) and \( q \geq 1 \) such that

\[ \beta_1 [(t + 1)^{p-1} - (t + 1)^{-1-q}] \leq \delta''(t) \leq \beta_2 [(t + 1)^{p-1} - (t + 1)^{-1-q}] \text{ for } t > 0. \]  

(20)

Note that condition (20) actually implies the strict convexity of an \( S^\ast \)-regular function on \((0, \infty)\). In particular, setting \( \beta_1 = \beta_2, \) condition (20) reduces to an equation

\[ \delta''(t) = (t + 1)^{p-1} - (t + 1)^{-1-q}. \]  

(21)

Taking integration twice and noting that \( \lim_{t \to 0^+} \delta(0) = \lim_{t \to 0^+} \delta'(0) = 0 \), the unique solution to equation (21) is

\[ \Delta_{p,q}(t) = \frac{(t + 1)^{p+1} - 1}{p(p + 1)} - \frac{(t + 1)^{1-q} - 1}{q(q - 1)} - \frac{p + q}{pq} t \text{ for } p \geq 1 \text{ and } q > 1. \]  

(22)

In addition, since \( \lim_{q \to 1^+} [1 - (t + 1)^{1-q}]/(q - 1) = \ln(t + 1) \), we have

\[ \Delta_{p,1}(t) = \frac{(t + 1)^{p+1} - 1}{p(p + 1)} + \ln(t + 1) - \frac{p + 1}{p} t \text{ for } p \geq 1. \]  

(23)

Taking \( p = 1 \) in (23), we have

\[ \Delta_{1,1}(t) = \frac{(t + 1)^2 - 1}{2} + \ln(t + 1) - 2t = \frac{1}{2} t^2 - t + \ln(t + 1). \]  

(24)

Moreover, taking \( p = 1 \) and \( q = 2 \) in (22) yields

\[ \Delta_{1,2}(t) = \frac{1}{2} \left[ (t + 1)^2 - (t + 1)^{-1} - 3t \right]. \]  

(25)
In terms of this particular solution \( \Delta_{p,q}(t) \), condition (20) can be written as

\[
\beta_1 \Delta''_{p,q}(t) \leq \delta''(t) \leq \beta_2 \Delta''_{p,q}(t). \tag{26}
\]

By integrating and noting that \( \lim_{t \to 0^+} \delta'(0) = \lim_{t \to 0^+} \delta(0) = 0 \), we further have

\[
\beta_1 \Delta'_p(t) \leq \delta'(t) \leq \beta_2 \Delta_p(t) \tag{27}
\]

and

\[
\beta_1 \Delta_{p,q}(t) \leq \delta(t) \leq \beta_2 \Delta_{p,q}(t). \tag{28}
\]

Therefore, we can see that the class of \( S^* \)-regular functions is quite broad. Later, by using inequalities (26)-(28), we show that \( S^* \)-regular functions fall into the category \( U_3 \).

It is worth mentioning that for any \( p \geq 1, q > 1 \) (including the case of \( q \to 1^+ \)) the \( S^* \)-regular function \( \Delta_{p,q}(t) \) is not self-concordant. In fact, the function \( \Delta_{p,q}(t) \) does not satisfy the inequality (17) since \( \delta''(t) \to 0 \) and \( \delta'''(t) \to p + q \) as \( t \to 0^+ \).

The \( S^* \)-regular functions are somewhat analogous to (but different from) the so-called self-regular functions that were defined in [18] to study interior-point algorithms for linear optimization.

**Definition 3.2.** [18] A twice differentiable function \( \psi(t) : (0, \infty) \to R \) is self-regular if (i) \( \psi(t) \) is strictly convex and vanishes at its global minimal point \( t = 1 \), i.e., \( \psi(1) = \psi'(1) = 0 \), and (ii) there exist constants \( \nu_2 \geq \nu_1 > 0 \) and \( p \geq 1, q \geq 1 \) such that

\[
\nu_1(t^{p-1} + t^{-1-q}) \leq \psi''(t) \leq \nu_2(t^{p-1} + t^{-1-q}) \quad \text{for} \quad t \in (0, \infty). \tag{29}
\]

Notice that a self-regular function is not necessarily increasing. However, a translation of variable leads to a strictly increasing function \( g(t) := \psi(t+1) \). We call such \( g(t) \) a translated self-regular function (\( T^* \)-regular in short). From Definition 3.2, we see that a \( T^* \)-regular function \( g(t) \) satisfies the following three conditions: (i) \( g(t) \) is strictly increasing on \( (0, \infty) \), (ii) \( \lim_{t \to 0^+} g(t) = \lim_{t \to 0^+} g'(t) = 0 \), and (iii) there exists positive constants \( \nu_2 \geq \nu_1 > 0 \) and \( p \geq 1, q \geq 1 \) such that

\[
\nu_1((t+1)^{p-1} + (t+1)^{-1-q}) \leq g''(t) \leq \nu_2((t+1)^{p-1} + (t+1)^{-1-q}) \quad \text{for} \quad t \in (0, \infty). \tag{29}
\]

Since \( \lim_{t \to 0^+} g''(t) \neq 0 \), \( T^* \)-regular functions and \( S^* \)-regular functions belong to two different classes, although conditions (29) and (20) may look alike. In fact, an \( S^* \)-regular function is not \( T^* \)-regular, and a \( T^* \)-regular function is not \( S^* \)-regular.

As pointed out in [18], when \( \nu_1 = \nu_2 \), the self-regular function \( \psi(t) \) is given by

\[
\psi_{p,q}(t) := \frac{t^{p+1}}{p(p+1)} + \frac{t^{1-q}}{q(q-1)} + \frac{p-q}{pq} (t-1) \quad \text{for} \quad p \geq 1 \quad \text{and} \quad q > 1. \tag{30}
\]
For $T^*$-regular function $G_{p,q}(t) := \psi_{p,q}(t + 1)$, condition (29) can be written as
\[ \nu_1 G_{p,q}''(t) \leq g''(t) \leq \nu_2 G_{p,q}''(t). \] (31)

Similar to (26) and (27), we have
\[ \nu_1 G_{p,q}'(t) \leq g'(t) \leq \nu_2 G_{p,q}'(t) \] (32)

and
\[ \nu_1 G_{p,q}(t) \leq g(t) \leq \nu_2 G_{p,q}(t). \] (33)

To compare with the $S^*$-regular functions $\Delta_{p,q}(t)$ (see, (22)-(25)), we list a few $T^*$-regular functions here:

\[ G_{p,q}(t) = \frac{(t+1)^{p+1} - 1}{p(p+1)} + \frac{(t+1)^{1-q} - 1}{q(q-1)} + \frac{p - q}{pq} t \quad \text{for } p \geq 1 \text{ and } q > 1. \]
\[ G_{p,1}(t) = \frac{(t+1)^{p+1} - 1}{p(p+1)} - \ln(t+1) + \frac{p-1}{p} t. \]
\[ G_{1,1}(t) = \frac{(t+1)^2 - 1}{2} - \ln(t+1) = \frac{1}{2} t^2 + (t - \ln(t+1)). \]
\[ G_{1,2}(t) = \frac{1}{2} \left[ (t+1)^2 + (t+1)^{-1} - t - 2 \right]. \]

As pointed out in [18], the self-regular function (30) is self-concordant [16]. Thus, the $T^*$-regular function $G_{p,q}(t)$ can be also viewed as a translated self-concordant function.

While $T^*$-regular and $S^*$-regular functions belong to two different classes, the following theorem shows that they both belong to the category of $\mathcal{U}_3$.

**Theorem 3.2.** Let $\delta(t) : (0, \infty) \to R$ be either $S^*$-regular or $T^*$-regular on $(0, \infty)$. Then there exist $c_2 > c_1 > 0$ such that
\[ c_1 \leq \frac{\delta(t)\delta''(t)}{[\delta'(t)]^2} \leq c_2 \text{ for } t \in (0, \infty), \] (34)

i.e., the function $\delta(t) \in \mathcal{U}_3$.

**Proof.** We first show that an $S^*$-regular function $\Delta_{p,q}(t)$ satisfies the property (34). Actually, we have
\[ \frac{\Delta_{p,q}(t)\Delta_{p,q}''(t)}{[\Delta_{p,q}(t)]^2} = \frac{(t+1)^{p+1} - 1}{p(p+1)} - \frac{(t+1)^{1-q} - 1}{q(q-1)} - \frac{p+q}{pq} t \frac{[(t+1)^{p-1} - (t+1)^{-1-q}]}{\left(\frac{t+1}{p} + \frac{(t+1)^{-q}}{q} - \frac{p+q}{pq}\right)^2}. \]

Dividing the numerator and denominator of the right-hand side of the above equation by $(t+1)^{2p} = (t+1)^{p+1}(t+1)^{p-1}$, we have
\[ \frac{\Delta_{p,q}(t)\Delta_{p,q}''(t)}{[\Delta_{p,q}(t)]^2} = \frac{\frac{1}{p} + \frac{1}{q(t+1)^{p+q}} - \frac{p+q}{pq(t+1)^p}}{\left(\frac{1}{p} + \frac{1}{q(t+1)^{p+q}} - \frac{p+q}{pq(t+1)^p}\right)^2}. \]
Therefore,
\[
\lim_{t \to \infty} \frac{\Delta_{p,q}(t)\Delta''_{p,q}(t)}{[\Delta_{p,q}(t)]^2} = \frac{p}{p + 1}.
\]  
(35)

Notice that \(\Delta''_{p,q}(t) = (t + 1)^{p-1} - (t + 1)^{1-q}\). We have
\[
\lim_{t \to 0^+} \Delta''_{p,q}(t) = \lim_{t \to 0^+} (p - 1)(t + 1)^{p-2} + (1 + q)(t + 1)^{-2-q} = p + q.
\]

Since \(\Delta''_{p,q}(t) \to 0\), \(\Delta'_{p,q}(t) \to 0\) and \(\Delta_{p,q}(t) \to 0\) as \(t \to 0^+\), we have
\[
\lim_{t \to 0^+} \frac{(\Delta''_{p,q}(t))^2}{\Delta'_{p,q}(t) \Delta''_{p,q}(t)} = \lim_{t \to 0^+} \frac{[\Delta''_{p,q}(t)]^2}{[\Delta'_{p,q}(t)]^2} = \lim_{t \to 0^+} \frac{2\Delta''_{p,q}(t)\Delta'''_{p,q}(t)}{\Delta''_{p,q}(t)} = 2(p + q).
\]

Hence
\[
\lim_{t \to 0^+} \frac{\Delta_{p,q}(t)}{2\Delta'_{p,q}(t) \Delta''_{p,q}(t)} = \lim_{t \to 0^+} \frac{\Delta_{p,q}(t)'}{2\Delta'_{p,q}(t) \Delta''_{p,q}(t)'} = \lim_{t \to 0^+} \frac{\Delta_{p,q}(t)'}{2\Delta''_{p,q}(t)^2 + 2\Delta'_{p,q}(t) \Delta'''_{p,q}(t)} = \frac{1}{2(2(p + q) + (p + q))} = \frac{1}{6(p + q)}.
\]

Using the above relations, we further have
\[
\lim_{t \to 0^+} \frac{\Delta_{p,q}(t)\Delta''_{p,q}(t)}{[\Delta'_{p,q}(t)]^2} = \lim_{t \to 0^+} \frac{\Delta_{p,q}(t)\Delta''_{p,q}(t)}{[\Delta'_{p,q}(t)]^2} = \lim_{t \to 0^+} \frac{\Delta_{p,q}(t)\Delta''_{p,q}(t) + \Delta_{p,q}(t)\Delta'''_{p,q}(t)}{2\Delta'_{p,q}(t) \Delta''_{p,q}(t)}
\]
\[
= \frac{1}{2} + \lim_{t \to 0^+} \frac{\Delta_{p,q}(t)\Delta'''_{p,q}(t)}{2\Delta'_{p,q}(t) \Delta''_{p,q}(t)}
\]
\[
= \frac{1}{2} + \lim_{t \to 0^+} \frac{\Delta_{p,q}(t)}{2\Delta'_{p,q}(t) \Delta''_{p,q}(t)} \lim_{t \to 0^+} \Delta'''_{p,q}(t)
\]
\[
= \frac{1}{2} + \frac{1}{6(p + q)}(p + q)
\]
\[
= \frac{2}{3}.
\]  
(36)

With (35) and (36), and notice that \(\Delta_{p,q}(t) > 0\), \(\Delta''_{p,q}(t) > 0\) and \(\Delta'_{p,q}(t) > 0\) in \((0, \infty)\), by continuity, we conclude that there exist two constants \(\mu_2 \geq \mu_1 > 0\) such that
\[
\mu_1 \leq \frac{\Delta_{p,q}(t)\Delta''_{p,q}(t)}{[\Delta'_{p,q}(t)]^2} \leq \mu_2 \quad \text{for } t \in (0, \infty).
\]
Together with (26) through (28), this implies that an $S^*$-regular function $\delta(t)$ satisfies the following inequality:

$$0 < \mu_1\beta_1 \leq \frac{\delta(t)\delta''(t)}{[\delta'(t)]^2} \leq \beta_2\mu_2,$$

Therefore, (34) holds with $c_1 := \mu_1\beta_1$ and $c_2 := \mu_2\beta_2$.

Now, for a $T^*$-regular function $G_{p,q}(t)$, we have

$$\frac{G_{p,q}(t)G''_{p,q}(t)}{[G'_{p,q}(t)]^2} = \frac{((t + 1)^{p+1} - (t + 1)^{q-1} + \frac{p-q}{pq}t)((t + 1)^{p+1} + (t + 1)^{1-q})}{\left((t + 1)^{p} - \frac{(t+1)^{1-q}}{q} + \frac{p-q}{pq}\right)^2}.$$ 

Dividing the numerator and denominator of the right-hand side of the above equation by $(t + 1)^{2p} = (t + 1)^{p+1}(t + 1)^{p-1}$, and taking $t \to \infty$, we have

$$\lim_{t \to \infty} \frac{G_{p,q}(t)G''_{p,q}(t)}{[G'_{p,q}(t)]^2} = \frac{p}{p + 1}. \quad (37)$$

Notice that

$$\lim_{t \to 0^+} G''_{p,q}(t) = \lim_{t \to 0^+} (t + 1)^{p-1} + (t + 1)^{-q} = 2.$$ 

Since $G_{p,q}(t) \to 0$ and $G'_{p,q}(t) \to 0$ as $t \to 0^+$, thus,

$$\lim_{t \to 0^+} \frac{G_{p,q}(t)G''_{p,q}(t)}{[G'_{p,q}(t)]^2} = \lim_{t \to 0^+} \frac{G_{p,q}(t)}{[G'_{p,q}(t)]^2} \lim_{t \to 0^+} G''_{p,q}(t)$$

$$= \lim_{t \to 0^+} \frac{G_{p,q}(t)}{[G'_{p,q}(t)]^2} \lim_{t \to 0^+} G''_{p,q}(t)$$

$$= \lim_{t \to 0^+} \frac{G_{p,q}(t)}{2G_{p,q}(t)G''_{p,q}(t)} \lim_{t \to 0^+} G''_{p,q}(t)$$

$$= \frac{1}{2}. \quad (38)$$

With (37) and (38), and notice that $G_{p,q}(t), G''_{p,q}(t)$, and $G'_{p,q}(t)$ are positive on $(0, \infty)$, by continuity, we conclude that there exist $\lambda_2 \geq \lambda_1 > 0$ such that

$$\lambda_1 \leq \frac{G_{p,q}(t)G''_{p,q}(t)}{[G'_{p,q}(t)]^2} \leq \lambda_2 \quad \text{for } t \in (0, \infty).$$

This together with (31) through (33) implies that a $T^*$-regular function satisfies the following inequality:

$$0 < \lambda_1\nu_1 \leq \frac{g(t)g''(t)}{[g'(t)]^2} \leq \lambda_2\nu_2.$$ 

Hence (34) holds with $c_1 := \nu_1\lambda_1$ and $c_2 := \nu_2\lambda_2$. □

A final remark should be made here is that new functions in $\mathcal{U}_1$ or $\mathcal{U}_2$ can be constructed by using the basic operations (addition, multiplication, division and composition) on known functions.
Theorem 3.3.

(i) If $\phi : (0, \infty) \to (0, \infty)$, $\phi \in \mathcal{U}_1$ with $\alpha = \alpha_1$ and $\varphi : (0, \infty) \to (0, \infty)$, $\varphi \in \mathcal{U}_1$ with $\alpha = \alpha_2$, then $\phi + \varphi \in \mathcal{U}_1$ with $\alpha = 2 \max\{\alpha_1, \alpha_2\}$.

(ii) If $\phi : (0, \infty) \to (0, \infty)$, $\phi \in \mathcal{U}_1$ with $\alpha_1 \in (0, 1]$ and $\varphi : (0, \infty) \to (0, \infty)$, $\varphi \in \mathcal{U}_1$ with $\alpha_2 \in (0, 1]$, then the multiplicative function $\phi(t) \cdot \varphi(t) \in \mathcal{U}_1$ with $\alpha = 1$. Similarly, if $\phi \in \mathcal{U}_2$ with $\alpha_1 \geq 1$ and $\varphi \in \mathcal{U}_2$ with $\alpha_2 \geq 1$, then $\phi(t) \cdot \varphi(t) \in \mathcal{U}_2$ with $\alpha = 1$.

(iii) If $\phi : (0, \infty) \to (0, \infty)$, $\phi \in \mathcal{U}_2$ with $\alpha_1 \geq 1$ and $\varphi : (0, \infty) \to (0, \infty)$, $\varphi \in \mathcal{U}_1$ with $\alpha_2 \in (0, 1]$, then the function $\frac{\phi}{\varphi} \in \mathcal{U}_2$ with $\alpha = 1$. Similarly, if $\phi \in \mathcal{U}_1$ with $\alpha_1 \in (0, 1]$ and $\varphi \in \mathcal{U}_2$ with $\alpha_2 \geq 1$, then $\frac{\phi}{\varphi} \in \mathcal{U}_1$ with $\alpha = 1$.

(iv) Let $\varphi : (0, \infty) \to \Omega_1 \subset R$ and $\phi : \Omega_1 \to (0, \infty)$ be two convex functions. If $\phi \in \mathcal{U}_1$ with $\alpha > 0$, then the composite function $(\phi \circ \varphi)(t) = \phi(\varphi(t)) \in \mathcal{U}_1$ with the same constant $\alpha$.

Proof. Keep in mind that all functions in $\mathcal{U}_1$ and $\mathcal{U}_2$ are convex, twice differentiable, and strictly increasing. For (i), we note that $\alpha_1, \alpha_2, \phi(t)$ and $\varphi(t)$ are all nonnegative. Thus,

$$
(\phi'(t) + \varphi'(t))^2 \leq 2[(\phi'(t))^2 + (\varphi'(t))^2]
$$

$$
\leq 2\alpha_1 \phi''(t) + 2\alpha_2 \varphi(t) \varphi''(t)
$$

$$
\leq 2 \max\{\alpha_1, \alpha_2\} \alpha(t) \varphi''(t) + \alpha(t) \varphi''(t))
$$

$$
\leq 2 \max\{\alpha_1, \alpha_2\} \alpha(t) \varphi''(t) + \alpha(t) \varphi''(t) + \alpha(t) \varphi''(t)
$$

$$
= 2 \max\{\alpha_1, \alpha_2\} \alpha(t) \varphi''(t) + \alpha(t) \varphi''(t) + \alpha(t) \varphi''(t).
$$

This indicates that $\phi + \varphi \in \mathcal{U}_1$ with $\alpha = 2 \max\{\alpha_1, \alpha_2\}$. The proofs of statements (ii) and (iii) can be easily provided by noting that

$$
(\phi(t) \varphi(t))(\phi(t) \varphi(t))'' = (\phi(t) \varphi(t))\phi''(t) + \alpha(t) \varphi''(t)
$$

$$
= \phi(t) \phi''(t) + \phi(t) \phi''(t) + \phi(t) \phi''(t) + \phi(t) \phi''(t) + \phi(t) \phi''(t)
$$

$$
= (\phi(t) \phi''(t) - \varphi'(t))^2 \varphi^2(t) + \phi^2(t) \varphi''(t) - \varphi'(t)^2)
$$

and

$$
\frac{\phi(t) \varphi(t)}{\varphi(t)}(\frac{\phi(t) \varphi(t)}{\varphi(t)})'' = \frac{\phi(t) \phi''(t) \varphi^2(t) - \phi^2(t) \varphi''(t) \varphi(t) - \varphi(t) \phi'(t) \varphi'(t)}{\varphi^4(t)}
$$

$$
= \frac{\phi(t) \phi''(t) - \varphi''(t) - \varphi''(t) \varphi'(t) - \varphi(t) \phi'(t) \varphi(t) + \phi^2(t) \varphi''(t)}{\varphi^4(t)}
$$

$$
\frac{\phi(t) \varphi(t)}{\varphi(t)}(\frac{\phi(t) \varphi(t)}{\varphi(t)})'' = \frac{\phi(t) \phi''(t) - \varphi''(t) - \varphi''(t) \varphi'(t) - \varphi(t) \phi'(t) \varphi(t)}{\varphi^4(t)} + \left(\frac{\phi(t) \varphi(t)}{\varphi(t)}\right)^2.
$$

Statement (iv) can also be easily verified and we omit its proof. □.
4 Constructing convex generalized log-exp functions

In this section, we show by examples how to construct a convex generalized log-exp function $\Gamma_w(x)$. Theorem 2.3 tells us that it suffices to find functions satisfying the inequalities (11) and (13) and compare their $\alpha$ values. In the second half of Section 3, we have identified several classes functions which can be used as building blocks. The remaining task is to estimate the $\alpha$ values, or equivalently, to estimate the values of $c_1$ and $c_2$ in (34). Let us start with using the $S^*$-regular and $T^*$-regular functions with $p = 1$ and $q = 1, 2$ to estimate required $c_1$ and $c_2$.

Theorem 4.1. The $S^*$-regular functions $\Delta_{1,1}(t)$ and $\Delta_{1,2}(t)$ given by (24) and (25), respectively, satisfy condition (34) with $c_1 = \frac{1}{2}$ and $c_2 = \frac{2}{3}$, that is,

$$\frac{3}{2} \Delta_{1,1}(t) \Delta''_{1,1}(t) \leq \left[ \Delta'_{1,1}(t) \right]^2 \leq 2 \Delta_{1,1}(t) \Delta''_{1,1}(t)$$  \hspace{1cm} (39)

and

$$\frac{3}{2} \Delta_{1,2}(t) \Delta''_{1,2}(t) \leq \left[ \Delta'_{1,2}(t) \right]^2 \leq 2 \Delta_{1,2}(t) \Delta''_{1,2}(t)$$  \hspace{1cm} (40)

for $t \in (0, \infty)$.

Proof. We first consider $\Delta_{1,1}(t)$. For this function, we have

$$\Delta'_{1,1}(t) = \frac{t^2}{t+1}, \quad \Delta''_{1,1}(t) = \frac{t(t+2)}{(t+1)^2},$$

and hence

$$\zeta(t) := \frac{\Delta_{1,1}(t) \Delta''_{1,1}(t)}{[\Delta'_{1,1}(t)]^2} = \frac{(t+2) \left( \frac{1}{2} t^2 - t + \ln(t+1) \right)}{t^3}.$$

Clearly, we have

$$\zeta'(t) = \frac{(\frac{1}{2} t^2 - t + \ln(t+1))(-2t - 6) + (t+2)t^2}{t^4}.$$

This implies that its stationary (including the maximum or minimum) point $t_*$ on $(0, \infty)$, if exists, satisfies that following equality:

$$\frac{1}{2} t_*^2 - t_* + \ln(t_*) + 1 = \frac{(t_* + 2)t_*^2}{(2t_* + 6)(t_* + 1)}.$$

The corresponding value of $\zeta(t)$ becomes

$$\zeta(t_*) = \frac{(t_* + 2) \left( \frac{1}{2} t_*^2 - t_* + \ln(t_* + 1) \right)}{t_*^3} = \frac{(t_* + 2)^2}{(2t_* + 6)(t_* + 1)}.$$

Let

$$\kappa(t) = \frac{(t + 2)^2}{(2t + 6)(t + 1)} \quad \text{for } t \in (0, \infty)$$

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It is easy to verify that \( \lim_{t \to 0^+} \kappa(t) = \frac{2}{3} \), \( \lim_{t \to -\infty} \kappa(t) = \frac{1}{2} \), and \( \kappa'(t) < 0 \) for \( t \in (0, \infty) \). Thus \( \frac{1}{2} \leq \kappa(t) \leq \frac{2}{3} \) for \( t \in (0, \infty) \). Since \( \zeta(t_*) = \kappa(t_*) \), the extremum value of \( \zeta(t) \) on \( (0, \infty) \) is located in the interval \( \left[ \frac{1}{2}, \frac{2}{3} \right] \). By (35) and (36), we see that \( \zeta(t) \to \frac{2}{3} \) as \( t \to 0^+ \), \( \zeta(t) \to \frac{1}{2} \) as \( t \to \infty \). Hence, we have

\[
\frac{1}{2} \leq \zeta(t) \leq \frac{2}{3} \quad \text{for} \quad t \in (0, \infty).
\]

This validates inequality (39).

We now show that inequality (40) holds. By simple calculation, we have

\[
\Delta_{1,2}(t) = \frac{t^3}{2(t+1)}, \quad \Delta'_{1,2}(t) = \frac{t^2(2t + 3)}{2(t+1)^2}, \quad \Delta''_{1,2}(t) = \frac{t(t^2 + 3t + 3)}{(t+1)^3}.
\]

Therefore,

\[
\xi(t) := \frac{\Delta'_{1,2}(t) \Delta''_{1,2}(t)}{[\Delta'_{1,2}(t)]^2} = \frac{2t^2 + 6t + 6}{4t^2 + 12t + 9}.
\]

Notice that

\[
\lim_{t \to 0^+} \xi(t) = \frac{2}{3} \quad \text{and} \quad \lim_{t \to \infty} \xi(t) = \frac{1}{2}.
\]

It is easy to verify that \( \xi'(t) = -6/(2t+3)^3 < 0 \) for \( t \in (0, \infty) \). Therefore, \( \xi(t) \) is decreasing over \( (0, \infty) \). With (41), we have

\[
\frac{1}{2} \leq \xi(t) \leq \frac{2}{3}.
\]

This validates inequality (40). \( \Box \)

A similar result for the \( T^* \)-regular functions \( G_{1,1}(t) \) and \( G_{1,2}(t) \) can be obtained.

**Theorem 4.2.** The \( T^* \)-regular function \( G_{1,1}(t) \) satisfies condition (34) with \( c_1 = \frac{1}{8} \) and \( c_2 = \frac{1}{2} \), while \( G_{1,2}(t) \) satisfies condition (34) with \( c_1 = \frac{12}{25} \) and \( c_2 = \frac{1}{7} \), i.e.,

\[
2G_{1,1}(t)G''_{1,1}(t) \leq [G'_{1,1}(t)]^2 \leq \frac{9}{4} G_{1,1}(t)G''_{1,1}(t)
\]

and

\[
2G_{1,2}(t)G''_{1,2}(t) \leq [G'_{1,2}(t)]^2 \leq \frac{29}{12} G_{1,2}(t)G''_{1,2}(t)
\]

for \( t \in (0, \infty) \).

**Proof.** It is evident that

\[
G'_{1,1}(t) = \frac{t(t+2)}{t+1}, \quad G''_{1,1}(t) = \frac{t^2 + 2t + 2}{(t+1)^2},
\]

and hence

\[
\chi(t) := \frac{G_{1,1}(t)G''_{1,1}(t)}{[G'_{1,1}(t)]^2} = \frac{\left(\frac{1}{2}t^2 + t - \ln(t+1)\right)(t^2 + 2t + 2)}{(t^2 + 2t)^2}.
\]
It is not difficult to check that
\[ \chi'(t) = \frac{-(2t + 2)(t^2 + 2t + 4) \left( \frac{1}{2} t^2 + t - \ln(t + 1) \right) + \frac{(t^2 + 2t)^2(t + 2 + 2t)}{t+1}}{(2t + t^2)^3}. \]

The stationary point \( t_* \) of \( \chi(t) \) on \((0, \infty)\), if exists, must satisfy
\[ \frac{1}{2} t_*^2 + t_* - \log(t_* + 1) = \frac{(2t_* + t_*^2)(t_*^2 + 2t_* + 2)}{2(t_* + 1)^2(t_*^2 + 2t_* + 4)}. \]

Therefore, if the \( \chi(t) \) has an extremum point \( t_* \), its extremum value is given by
\[
\chi(t_*) = \frac{\left( \frac{1}{2} t_*^2 + t_* - \log(t_* + 1) \right)(t_*^2 + 2t_* + 2)}{(t_*^2 + 2t_*)^2} = \frac{(2t_* + t_*^2)(t_*^2 + 2t_* + 2)}{2(t_* + 1)^2(t_*^2 + 2t_* + 4)} \cdot \frac{(t_*^2 + 2t_* + 2)}{(t_*^2 + 2t_*)^2} = \frac{2(t_* + 1)^2}{[t_*^2 + 1]^2 + 3},
\]
where \( \omega(t) := \frac{(t+1)^2}{2(t+3)} \) for \( t \in (1, \infty) \). It is evident that \( \lim_{t \to 1^+} \omega(t) = \frac{1}{2} \) and \( \lim_{t \to \infty} \omega(t) = \frac{1}{2} \).

We can also check that \( \omega'(t) = 0 \) has a unique solution over \((1, \infty)\) at \( t_* = 3 \) with value \( \omega(t_*) = \frac{4}{9} \). Therefore, \( \frac{4}{9} \leq \omega(t) \leq \frac{1}{2} \) for \( t \in (1, \infty) \). This means that
\[
\frac{4}{9} \leq \chi(t_*) = \omega((t_* + 1)^2) \leq \frac{1}{2}.
\]

From (37) and (38), we see that \( \lim_{t \to 0^+} \chi(t) = \frac{1}{2} = \lim_{t \to \infty} \chi(t) \). Hence inequality (42) holds.

We now prove inequality (43). Notice that
\[
G_{1,2}(t) = \frac{t^2(t + 2)}{2(t + 1)^2}, \quad G'_{1,2}(t) = \frac{t(2t^2 + 5t + 4)}{2(t + 1)^2}, \quad G''_{1,2}(t) = \frac{t^3 + 3t^2 + 3t + 2}{(t + 1)^3}.
\]

Therefore,
\[
\rho(t) := \frac{G_{1,2}(t)G''_{1,2}(t)}{[G'_{1,2}(t)]^2} = \frac{2(t + 2)(t^3 + 3t^2 + 3t + 2)}{(2t^2 + 5t + 4)^2} = \frac{2t^4 + 10t^3 + 18t^2 + 16t + 8}{4t^3 + 20t^2 + 41t^2 + 40t + 16}.
\]

Obviously,
\[
\lim_{t \to 0^+} \rho(t) = \frac{1}{2} = \lim_{t \to \infty} \rho(t).
\]

It is not difficult to verify that
\[
\rho'(t) := \frac{10t^3 + 24t^2 - 16}{(2t^2 + 5t + 4)^3}.
\]
The extremum point of \( \rho(t) \) over \((0, \infty)\), if exists, must satisfy that \( \rho'(t) = 0 \). For any such extreme point \( t^* \), we have
\[
8 = 5t^*_3 + 12t^*_2. \tag{45}
\]
Its extremum value is
\[
\rho(t^*) = \frac{2t^*_3 + 10t^*_2 + 18t^*_2 + 16t_* + 8}{4t^*_3 + 20t^*_2 + 41t^*_2 + 40t_* + 16} = \frac{2t^*_3 + 10t^*_2 + 18t^*_2 + 16t_* + (5t^*_3 + 12t^*_2)}{4t^*_3 + 20t^*_2 + 41t^*_2 + 40t_* + 2(5t^*_3 + 12t^*_2)} = \frac{2t^*_3 + 15t^*_2 + 30t_* + 16}{4t^*_3 + 30t^*_2 + 65t_* + 40} = \frac{2t^*_3 + 15t^*_2 + 30t_* + 2(5t^*_3 + 12t^*_2)}{4t^*_3 + 30t^*_2 + 65t_* + 5(5t^*_3 + 12t^*_2)} = \frac{12t^*_2 + 39t_* + 30}{29t^*_2 + 90t_* + 65}.
\]

Let
\[
\varsigma(t) := \frac{12t^2 + 39t + 30}{29t^2 + 90t + 65}.
\]
Then it is easy to check that \( \varsigma'(t) < 0 \) for \( t \in (0, \infty) \). Hence the function \( \varsigma(t) \) is decreasing on \((0, \infty)\). Notice that
\[
\lim_{t \to 0^+} \varsigma(t) = \frac{30}{65} = \frac{6}{13}, \quad \lim_{t \to \infty} \varsigma(t) = \frac{12}{29}.
\]
Therefore,
\[
\frac{12}{29} \leq \varsigma(t) \leq \frac{6}{13} < \frac{1}{2} \text{ for } t \in (0, \infty).
\]
Since \( \rho(t^*) = \varsigma(t^*) \), it follows from (44) that
\[
\frac{12}{29} \leq \rho(t) \leq \frac{1}{2} \text{ for } t \in (0, \infty).
\]
Inequality (43) follows from the definition of \( \rho(t) \). \( \Box \)

The next result shows that the composition functions of \( e^t \) belong to the category \( \mathcal{U}_3 \).

**Theorem 4.3.** Denote the exponential function \( e^t \) by \( \exp(t) \) and the composition of \( m \) \((m \geq 1)\) exponential functions by
\[
\theta_m(t) := (\exp \circ \exp \circ \cdots \circ \exp)(t).
\]
Then
\[
\frac{1}{m} \theta_m(t) \theta_m''(t) \leq [\theta_m'(t)]^2 \leq \theta_m(t) \theta_m''(t) \text{ for } t \in R. \tag{46}
\]
Proof. Let \( \alpha_m(t) := [\theta_m'(t)]^2/((\theta_m(t)\theta_m''(t)) \) for \( t \in R \). Since \( \alpha_1(t) \equiv 1 \), we can prove the right-hand side inequality of (46) using (iv) of Theorem 3.3 and mathematical induction. For the left-hand side inequality, notice that
\[
\theta_m'(t) = \theta_m(t)\theta_m''(t), \quad \theta_m''(t) = \theta_m(t)\left(\theta_m'(t)\right)^2 + \theta_m(t)\theta_m''(t)
\] for \( t \in R \).
This indicates that
\[
\alpha_m(t) = \frac{1}{1 + \frac{1}{\alpha_m(t)\theta_m''(t)}} > \frac{1}{1 + \frac{1}{\alpha_m(t)}} \quad \text{for } t \in R.
\]
It is easy to check that \( \alpha_2(t) \in (\frac{1}{2}, 1) \). The desired result follows by induction. \( \square \).

We are now ready to show a mechanism that constructs some convex generalized log-exp functions \( \Gamma_w(x) \), other than those simple ones derived from Corollary 3.1. In this case, the inner and outer functions of \( \Gamma_w(x) \) may be different, and countless choices become available.

**Theorem 4.4.** Let \( \Omega \) be an open convex subset of \( R \).
(i) Let \( \phi : \Omega \to (0, \infty) \) be a convex, twice differentiable, strictly increasing function on \( \Omega \). If \( \phi(t)\phi''(t) \leq \lfloor \phi'(t) \rfloor^2 \) for \( t \in \Omega \), then the generalized log-exp function
\[
\Gamma_w(x) := \phi^{-1}\left(\sum_{i=1}^{n} \frac{w_i}{\phi(x_i)^r}\right)
\]
is convex on \( \Omega^n \) for any given \( w \in R_{++}^n \) and \( r > 0 \).
(ii) Let \( \kappa > 0 \) be a constant and \( \phi : \Omega \to (\kappa, \infty) \) be a convex, twice differentiable, strictly increasing function satisfying the inequality \( \phi(t)\phi''(t) \leq \lfloor \phi'(t) \rfloor^2 \) for \( t \in \Omega \). Then, for any given \( w \in R_{++}^n \) and \( T > 0, r > 0 \), the generalized log-exp function
\[
\Gamma_w(x) = \ln \circ \ln \circ \cdots \circ \ln \left(\sum_{i=1}^{n} w_i \left(T + \frac{1}{\phi(x_i)^r}\right)\right)
\]
is convex on \( \Omega^n \) for any positive integer \( \ell \leq T\kappa^r + 1 \).

Proof. Result (i) comes from part (A) of Theorem 2.4 and Theorem 2.3. Result (ii) follows from Theorems 4.3 and 2.3, and part (B) of Theorem 2.4. In fact, it suffices to take the inner function \( h_T(t) = T + \frac{1}{\phi(t)^r} \) and outer function \( \theta_m(t) \), as defined in Theorem 4.3, whose inverse function is given by \( \ln \circ \ln \circ \cdots \circ \ln(t) \). \( \square \)

Using Theorems 2.3, 2.4, 4.1, 4.2 and 4.4, we see the generalized log-exp functions listed below in Example 4.1 are convex.

**Example 4.1.**

(i) \( \Delta_{1,1}^{-1} \left[ \sum_{i=1}^{n} \frac{1}{\Delta_{1,1}(x_i)^r} \right], \Delta_{1,2}^{-1} \left[ \sum_{i=1}^{n} \frac{1}{\Delta_{1,2}(x_i)^r} \right], G_{1,1}^{-1} \left[ \sum_{i=1}^{n} \frac{1}{G_{1,1}(x_i)^r} \right], G_{1,2}^{-1} \left[ \sum_{i=1}^{n} \frac{1}{G_{1,2}(x_i)^r} \right] \).
(ii) \(\ln \left( \sum_{i=1}^{n} \frac{1}{\Delta_{1,1}(x_i)^r} \right), \ln \left( \sum_{i=1}^{n} \frac{1}{\Delta_{1,2}(x_i)^r} \right), \ln \left( \sum_{i=1}^{n} \frac{1}{G_{1,1}(x_i)^r} \right), \ln \left( \sum_{i=1}^{n} \frac{1}{G_{1,2}(x_i)^r} \right).\)

(iii) \(\ln \left( \sum_{i=1}^{n} (m + e^{-rx_i}) \right), \ldots, \ln \circ \ln \circ \cdots \ln \left( \sum_{i=1}^{n} (m + e^{-rx_i}) \right) \text{ on } (0, \infty)^n.\)

(iv) \(\ln \left[ \sum_{i=1}^{n} \left( m + \frac{1}{\Delta_{1,1}(x_i)^r} \right) \right], \ldots, \ln \circ \ln \circ \cdots \ln \left[ \sum_{i=1}^{n} \left( m + \frac{1}{\Delta_{1,1}(x_i)^r} \right) \right] \text{ on } (\tau, \infty)^n\)

with \(\ell \leq m\Delta_{1,1}(\tau)^r + 1\) and \(\tau > 0.\)

It follows from Corollary 3.1 that the function \(x^p\) over \((0, \infty)\) satisfies the equation (16) with \(\alpha = \frac{p}{p-1}\). When \(1 < p \leq 2\), \(\alpha \geq 2\) and when \(1 < p \leq \frac{29}{17}\), \(\alpha \geq \frac{29}{12} \geq \frac{9}{4}\). By Theorems 4.1 and 4.2, we know that both \(\Delta_{1,2}(t)\) and \(\Delta_{1,1}(t)\) satisfy condition (13) with \(\alpha = 2\), and both \(G_{1,2}(t)\) and \(G_{1,1}(t)\) satisfy condition (13) with \(\alpha = \frac{29}{12}\). From Theorem 2.3, we see the functions listed in Example 4.2 are convex.

Example 4.2.

(a) Let \(1 < p \leq 2\) and \(\delta_i(t) = \Delta_{1,2}(t)\) or \(\Delta_{1,1}(t)\), for \(t \in (0, \infty)\) and \(i = 1, \ldots, n\). Then \(\Gamma_w(x) = \left( \sum_{i=1}^{n} w_i \delta_i(x_i) \right)^{\frac{1}{p}}\) is convex on \((0, \infty)^n\).

(b) Let \(1 < p \leq \frac{29}{17}\) and \(\delta_i(t)\) be \(G_{1,2}(t), G_{1,1}(t), \Delta_{1,2}(t)\) or \(\Delta_{1,1}(t)\), for \(t \in (0, \infty)\) and \(i = 1, \ldots, n\). Then \(\Gamma_w(x) = \left( \sum_{i=1}^{n} w_i \delta_i(x_i) \right)^{\frac{1}{p}}\) is convex on \((0, \infty)^n\).

Example 4.3.

(a) Let \(\delta_i(t), i = 1, \ldots, n,\) be \(\Delta_{1,1}(t)\) or \(\Delta_{1,2}(t)\) for \(t \in (0, \infty)\). Then \(\Gamma_w(x) = G_{1,1}^{-1} \left( \sum_{i=1}^{n} w_i \delta_i(x_i) \right)\) is convex on \((0, \infty)^n\).

(b) Let \(\delta_i(t), i = 1, \ldots, n,\) be \(\Delta_{1,1}(t)\) or \(\Delta_{1,2}(t)\) for \(t \in (0, \infty)\). Then \(\Gamma_w(x) = G_{1,2}^{-1} \left( \sum_{i=1}^{n} w_i \delta_i(x_i) \right)\) is convex on \((0, \infty)^n)\).

(c) Let \(\delta_i(t)\) be \(\Delta_{1,1}(t), \Delta_{1,2}(t), G_{1,1}(t)\) or \(G_{1,2}(t)\) for \(t \in (0, \infty)\). Then \(\Gamma_w(x) = \delta^{-1} \left( \sum_{i=1}^{n} w_i e^{x_i} \right)\) is convex on \(R^n\).

Remark 4.1. If we take the outer function \(\Psi(t)\) to be \(\Delta_{p,q}(t)\) or \(G_{p,q}(t)\), since both functions are strictly increasing, the inverse function \(\Psi^{-1}(y)\) exists and is unique. To find the inverse function, we may solve a polynomial equation \(\Delta_{p,q}(t) - y = 0\) or \(G_{p,q}(t) - y = 0\) with respect to \(y\). It is well known that any polynomial \(P(t)\) with degree less or equal to 4 has explicit solutions (see, for example, Borwein and Erdélyi [4]). Thus, the inverse function \(\Delta_{1,2}^{-1}(t)\) and \(G_{1,2}^{-1}(t)\) can be explicitly calculated by a formula containing additions, subtractions, multiplications, divisions and radicals. We may as well take \(\Psi(t)\) to be \(\Delta_{1,3}(t), \Delta_{2,2}(t), G_{1,3}(t)\), or \(G_{2,2}(t)\) to construct a generalized log-exp function, but it will be difficult to calculate its inverse function on \((0, \infty)\), because we have no explicit solution formula for polynomials with
degree more than 5. Thus, the way of construction as shown in Example 4.3 may only be applicable for polynomial with degree less than or equal to 4.

5 Final remarks

In this paper, we have extended the well known log-exp function (1) and the $p$-norms to a class of functions called generalized log-exp functions. We have also established some necessary and sufficient condition for this class of functions to be convex. Some useful and easy-to-test sufficient conditions have been developed for users to construct convex generalized log-exp functions. As a result, the aforementioned two fundamental questions (raised in Section 1) have been addressed. Moreover, a systematic way to explicitly construct convex generalized log-exp functions has been illustrated. To this end, the new definitions of $S^*$-regular and $T^*$-regular functions were introduced. The later is essentially a transformation of the self-regular function proposed in [18]. It should be noted that most of $S^*$-functions are not self-concordant, while the class of $T^*$-regular (self-regular) functions has a large overlap with that of self-concordant functions [16]. Self-concordant and self-regular functions are widely used in interior-point methods for convex programming and linear programming (see for example, [16, 18]).

It is worth mentioning a related result here. Consider the log-exp function again. Notice that it is a convex function by imposing one concave-transformation $y \rightarrow \ln y$ on the convex function $\sum_{i=1}^{n} e^{x_i}$. However, it is evident that if we impose double log-concave-transformations on it, the resulting function $\ln \ln(\sum_{i=1}^{n} e^{x_i})$ is no longer convex. Thus, an interesting mathematical question arises: Given a convex function, how many times of log-transformations can it take without losing convexity? Theorem 4.1 in this paper partially answers this question and indicates how to construct such a function.

It should be noted that the conjugate function of a generalized log-exp function, in a sense, can be viewed as a generalization of Shannon’s entropy function. For future work, we are interested in finding novel applications of the generalized log-exp functions and corresponding conjugates in mathematical optimization and other related fields.

References


