LECTURE 12: QUADRATICALLY CONSTRAINED QUADRATIC PROGRAMMING (QCQP)

1. Motivation
2. Convex QCQP
3. General QCQP
4. On-Going Research
Quadratically Constrained Quadratic Programming (QCQP)

- General Form:

\[
\text{QCQP:} \\
\min x^T A_0 x + 2b_0^T x + c_0 \\
\text{s.t. } x^T A_i x + 2b_i^T x + c_i \leq 0, \quad i = 1, \ldots, m_1, \\
\quad \quad \quad x^T A_i x + 2b_i^T x + c_i = 0, \quad i = m_1 + 1, \ldots, m_1 + m_2, \\
A_i \in S^n, \quad b_i \in \mathbb{R}^n, \quad c_i \in \mathbb{R}, \quad i = 0, 1, \ldots, m_1 + m_2.
\]

where

\(S^n\) is the set of symmetric square matrices of order \(n\).
Motivation

**QCQP** is a
- connection between LP and NLP problems.
- second-order approximation of nonlinearity.
- subroutine used in NLP solution methods.
- first step toward non-convex optimization.
- bridge to binary integer programming.
- generalization of commonly used models such as linear regression and generalized eigenvalue problems.

Fact: **QCQP is in general NP-Hard.**
Why are QCQP problems important?

- **Wide Applications**
  - Portfolio Optimization Problem
  - Knapsack Problem
  - Location-allocation Problem
  - Information Network Security
  - Combinatorial Problems
  - Graph Theory
  . . .
Why are QCQP problems important?

- Generalizations of known optimization problems
  - Standard Quadratic Programming Problem
  - 0-1 Quadratic Programming Problem
  - Box Constrained Quadratic Programming Problem
  - Mixed-integer Quadratic Programming Problem
  - Second-order Cone Constrained Quadratic Programming Problem
  - Co-positive Programming Problem
  ...

Application I: Portfolio Optimization Problem

What is the best portfolio selection for your investment?

Invest instead of saving in your pocket!
Application I: Portfolio Optimization Problem

The classical mean-variance (MV) model developed by Markowitz (1952) uses mean and variance of the portfolio to measure the expected value and risk of the selection.
Let \( x = (x_1, x_2, \ldots, x_n) \) be the vector of portfolio weights investing on \( n \) securities.
Let \( \xi \) be the random vector of expected returns of \( n \) risky assets.
\[
\mu_i = E(\xi_i), \quad i = 1, \ldots, n, \quad \sigma^2(\xi^T x) = x^T Q x.
\]

\[
\begin{align*}
\min & \quad x^T Q x \\
\text{s.t.} & \quad \mu^T x \geq \rho, \\
& \quad A x \leq b,
\end{align*}
\]

(MV)

where \( \rho \) is a prescribed return level, \( Ax \leq b \) is used for representing some real-world trading conditions.
Application I: Portfolio Optimization Problem with Hard Constraints

- **Cardinality constraint**: the number of assets in the optimal portfolio should be limited,

\[ |\text{supp}(x)| \leq K, \]

where \( \text{supp}(x) = \{i \mid x_i \neq 0\}, 1 \leq K << n. \)

The need to account for this limit is due to the transaction cost and managerial concerns.

- **Minimum buy-in threshold**:

\[ \alpha_i \leq x_i \leq \beta_i, \ i \in \text{supp}(x) \quad \text{or} \quad x_i \in \{0\} \cup [\alpha_i, \beta_i], \ i = 1, \ldots, n. \]

**Difficulty**: testing the feasibility of the domain defined by the new constraints is already NP-complete when A has three rows, see Bienstock(1996).
Application I: Reformulation of Portfolio Optimization Problem as QCQP

- The cardinality constraint can be represented by:
  \[ e^T y \leq K, \quad 0 \leq x_i \leq \beta_i y_i, \quad i = 1, \ldots, n, \quad y \in \{0, 1\}^n. \]

- The minimum buy-in threshold can be expressed as:
  \[ \alpha_i y_i \leq x_i \leq \beta_i y_i, \quad i = 1, \ldots, n, \quad y \in \{0, 1\}^n. \]

- Mixed-integer constrained quadratic programming (MIQP) problem:

\[
\begin{align*}
\min & \quad x^T Q x \\
\text{s.t.} & \quad Ax \leq b, \quad e^T y \leq K, \\
& \quad y_i^2 - y_i = 0, \quad (y_i \in \{0, 1\}), \quad i = 1, \ldots, n, \\
& \quad \alpha_i y_i \leq x_i \leq \beta_i y_i, \quad i = 1, \ldots, n.
\end{align*}
\]  

(CMV)
Application II: Quadratic Knapsack Problem

Which items will you pick for your weight-limited bag?
Application II: Quadratic Knapsack Problem

- Given $n$ items, where item $j$ has a positive integer weight $w_j$.

- Given a $n \times n$ nonnegative integer matrix $Q$, where $Q_{ii}$ is the profit achieved if item $i$ is selected and $Q_{ij} = Q_{ji}$ is the profit achieved if both items $i$ and $j$ are selected.

- Quadratic knapsack problem selects an item subset whose overall weight does not exceed a given knapsack capacity $c$, so as to maximize the overall profit.
Application II: Quadratic Knapsack Problem as QCQP

Let $w^T = (w_1, w_2, \ldots, w_n)$, by introducing the binary variable, the problem can be reformulated as

$$\begin{align*}
\text{max} & \quad x^TQx \\
\text{s.t.} & \quad w^T x \leq c, \\
& \quad x_i \in \{0, 1\}, i = 1, \ldots, n. \\
& \quad (x_i^2 - x_i = 0)
\end{align*}$$

(QKP)

Difficulty: quadratic knapsack problem is NP-Hard, see Gallo et al. (1980).

Application III: Location-Allocation Problem

Where to locate these factories?
Application III: Location-Allocation problem

- Given the flow $f_{ij}$ between facility $i$ and $j$, the distance $d_{kp}$ between location $k$ and $p$, for $i, j, k, p = 1, \ldots, n$.

- Assigning facilities to locations in such a way that each facility is designated to exactly one location and vice-versa.

- The location-allocation problem aims to find a minimum cost allocation of facilities into locations, taking the costs as the sum of all possible distance-flow products.

- The location-allocation problem is equivalent to the quadratic assignment problem.
Application III: Location-Allocation Problem as QCQP

\[ \min \sum_{i,j=1}^{n} \sum_{k,p=1}^{n} f_{ij}d_{kp}x_{ik}x_{jp} \]

\[ \sum_{i=1}^{n} x_{ij} = 1, \; j = 1, \ldots, n, \]

\[ \sum_{j=1}^{n} x_{ij} = 1, \; i = 1, \ldots, n, \]

\[ x_{ij} \in \{0, 1\}, \; i, j = 1, \ldots, n. \]

\[ (x_{ij}^2 - x_{ij} = 0) \]  

(QLAP)

Difficult: quadratic assignment problem is NP-Hard, see Sahni and Gonzales (1976).

Application IV: Information Network Security

For a hacker, what is the biggest damage to the information flow by destroying some edges in the network?

The problem of information network security is equivalent to the max-cut problem.
Application IV: Information Network Security as QCQP

- Given the weight $w_{ij} = w_{ji}$ for the edge between node $i$ and $j$.
- Introduce binary variable $x_i \in \{-1, 1\}$, $i = 1, \ldots, n$ to indicate the partition.

$$\max \quad \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \frac{1-x_i x_j}{2}$$

$$\text{s.t.} \quad x_i \in \{-1, 1\}, \quad i = 1, \ldots, n.$$  (MC)

$$\quad (x_i^2 = 1)$$

- **Difficulty**: max-cut problem is NP-Complete, see Karp(1972).

- If $w_{ij} \geq 0$, $\forall i \neq j$. Then the expected value of randomized algorithm is at least $\alpha \approx 0.878$ times the value of the maximum cut by using Semidefinite Programming, see Goemans and Williamson(1995).

Known Facts about QCQP

• QCQP: Kuhn and Tucker (1951).

• QP is **NP-hard**: even if \( A_0 \) has only one negative eigenvalue (Pardalos and Vavasis (1991)).

• **Convex QCQP**: Computable by interior point method (Nesterov and Nemirovskii (1994)).

• QCQP with **one quadratic constraint**: Computable when Slater’s condition holds (Sturm and Zhang (2003)); **Generalized Trust Region Problem** (Jin, Fang and Xing (2010))
Known Facts about QCQP

• QCQP with one ball and two parallel cutting planes: Computable (Burer (2011)).

• QCQP with two ellipsoid constraints: Only special cases are known to be computable.

• QCQP with one second order cone and two special linear constraints: Computable (Jin, Tian, Deng, Fang, Xing (2013)).
Current Research Directions

- Study the structure of QCQP problems when the number of constraints is small.
  - Identify polynomial-time solvable subclasses of QCQP problems

- Investigate sufficient conditions to ensure a KKT solution to be globally optimal.

- Propose a new computational scheme for designing efficient algorithms in solving some classes of QCQP problems.
  - Develop approximations for some difficult QCQP problems.
New Tool for QCQP: Linear Conic Programming

\[(LP)\]
\[
\begin{align*}
\text{min} & \quad c^T x \\
\text{s.t.} & \quad (a^i)^T x = b_i, \quad i = 1, \ldots, m, \\
& \quad x \geq 0. \\
& \quad (x \in \mathbb{R}^n_+) 
\end{align*}
\]

\[(LCoP)\]
\[
\begin{align*}
\text{min} & \quad C \cdot X \\
\text{s.t.} & \quad A_i \cdot X = b_i, \quad i = 1, \ldots, m, \\
& \quad X \in K. \\
& \quad (K \text{ is a cone})
\end{align*}
\]

\(K\) is a closed, convex cone; \(C, A_i\) and \(b_i\) are in the space of interests with “.” being an appropriate linear operator.
Special Cases of Linear Conic Programming

- Linear Programming (LP):
  \[ K = \mathbb{R}^n_+ \]

- Second-order Cone Programming (SOCP):
  \[ K = \mathcal{L}^n = \{ x \in \mathbb{R}^n \mid \sqrt{x_1^2 + \ldots + x_{n-1}^2} \leq x_n \} \]

- Semidefinite Programming (SDP):
  \[ K = S^n_+ = \{ M \in S^n \mid x^T M x \geq 0, \forall x \in \mathbb{R}^n \} \]

- Copositive Programming (CoP):
  \[ K = C^n = \{ M \in S^n \mid x^T M x \geq 0, \forall x \in \mathbb{R}^n_+ \} \]
Power of Linear Conic Programming

- Shares a very similar structure as Linear Programming.
- Nonlinearity and nonconvexity may be absorbed by the cone $K$.
- Possesses well-developed theory.
  - Optimality
  - Duality
  - Sensitivity
  - Polynomial-time interior-point algorithms
Computable LCoP

• Among LCoP problems, LP, SCOP and SDP exhibit good duality theorems.

• These three classes are computable with effective interior-point algorithms and commercially available software.

• They can also be used to provide approximations to non-computable problems.
Linear Program (LP)

\[
\begin{align*}
\text{Min} & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq_{\mathbb{R}^n} 0
\end{align*}
\]  
(LP)

\[
\begin{align*}
\text{Max} & \quad b^T y \\
\text{s.t.} & \quad A^T y + s = c \\
& \quad s \geq_{\mathbb{R}^n} 0
\end{align*}
\]  
(LD)
Linear Program (LP)

Theorem (LP duality theorem)

(i) If either LP or LD is unbounded, then the other one is infeasible.

(ii) If either $v(\text{LP})$ or $v(\text{LD})$ is finite, then there exist $x^* \in \text{feas}(\text{LP})$ and $(y^*, s^*) \in \text{feas}(\text{LD})$ such that $v(\text{LP}) = c^T x^* = b^T y^* = v(\text{LD})$.

(iii) If LP is feasible and $v(\text{LP})$ is finite, then $x^*$ is optimal for LP if and only if the following conditions hold:

(a) $Ax^* = b$, $x^* \succeq_{\mathbb{R}_+^n} 0$;

(b) there exists $(y^*, s^*)$ satisfying $A^T y^* + s^* = c$, $s \succeq_{\mathbb{R}_+^n} 0$;

(c) $(x^*)^T s^* = c^T x^* - b^T y^* = 0$. 
Second Order Cone Program (SOCP)

\[
\begin{align*}
    \text{Min} & \quad c^T x \\
    \text{s.t.} & \quad Ax = b \\
    & \quad x \geq_K 0
\end{align*}
\]  \quad \text{(SOCP)}

where \( K = \mathcal{L}_{n_1} \times \cdots \times \mathcal{L}_{n_r} = \{ x \in \mathbb{R}^n | n_1 + \cdots + n_r = n, (x_1, \ldots, x_{n_1})^T \in \mathcal{L}_{n_1}, \ldots, (x_{n-n_r+1}, \ldots, x_n)^T \in \mathcal{L}_{n_r} \} \).

\[
\begin{align*}
    \text{Max} & \quad b^T y \\
    \text{s.t.} & \quad A^T y + s = c \\
    & \quad s \geq_K 0
\end{align*}
\]  \quad \text{(SOCD)}
Second Order Cone Program (SOCP)

Theorem (SOCP duality theorem)

(i) If either SOCP or SOCD is unbounded, then the other one is infeasible.

(ii) If there exists a feasible solution $\bar{x}$ such that $\bar{x} \in \text{int}(K)$, and $v(\text{SOCP})$ is finite, then there exist $(y^*, s^*) \in \text{feas}(\text{SOCD})$ such that $v(\text{SOCP}) = b^T y^* = v(\text{SOCD})$.

(iii) If there exists a feasible solution $(\bar{y}, \bar{s})$ such that $\bar{s} \in \text{int}(K)$, and $v(\text{SOCD})$ is finite, then there exist $x^* \in \text{feas}(\text{SOCP})$ such that $v(\text{SOCP}) = c^T x^* = v(\text{SOCD})$. 
Second Order Cone Program (SOCP)

Theorem (SOCP duality theorem)

(iv) If both SOCP and SOCPD are feasible, and there exists a feasible solution $\bar{x}$ such that $\bar{x} \in \text{int}(K)$, then $x^*$ is optimal for SOCP if and only if the following conditions hold:

(a) $Ax^* = b$, $x^* \geq_K 0$;
(b) there exists $(y^*, s^*)$ satisfying $A^Ty^* + s^* = c$, $s^* \geq_K 0$;
(c) $(x^*)^Ts^* = c^Tx^* - b^Ty^* = 0$. 
Semidefinite Program (SDP)

\[
\begin{align*}
\text{Min} & \quad C \cdot X \\
\text{s.t.} & \quad AX = b \\
& \quad X \succeq 0 \\
\text{Max} & \quad b^T y \\
\text{s.t.} & \quad A^*y + S = C \\
& \quad S \succeq 0
\end{align*}
\]

(SDP) \quad \text{(SDD)}

Note:

\[
A^*y = \sum_{i=1}^{m} y_i A_i
\]
Semidefinite Program (SDP)

Theorem (SDP duality theorem)

(i) If either SDP or SDD is unbounded, then the other one is infeasible.

(ii) If there exists a feasible solution $\bar{X}$ such that $\bar{X} \succ 0$, and $v(\text{SDP})$ is finite, then there exist $(y^*, S^*) \in \text{feas}(\text{SDD})$ such that $v(\text{SDP}) = b^T y^* = v(\text{SDD})$.

(iii) If there exists a feasible solution $(\bar{y}, \bar{S})$ such that $\bar{S} \succ 0$, and $v(\text{SDD})$ is finite, then there exist $X^* \in \text{feas}(\text{SDP})$ such that $v(\text{SDP}) = C \cdot X^* = v(\text{SDD})$. 
Semidefinite Program (SDP)

Theorem (SDP duality theorem)

(iv) If both SDP and SDD are feasible, and there exists a feasible solution $\bar{X}$ such that $\bar{X} \succeq 0$, then $X^*$ is optimal for SDP if and only if the following conditions hold:

(a) $AX^* = b$, $X^* \succeq 0$;

(b) there exists $(y^*, S^*)$ satisfying $A^*y^* + S^* = C$, $S^* \succeq 0$;

(c) $X^* \bullet S^* = C \bullet X^* - b^T y^* = 0$. 
LCoP for QCQP

Quadratically Constrained Quadratic Programming (QCQP)

IQCQP:

\[
\begin{align*}
\text{Min} & \quad x^T A_0 x + 2b_0^T x + c_0 \\
\text{s.t.} & \quad x^T A_i x + 2b_i^T x + c_i \leq 0, \quad i = 1, \ldots, m \\
A_i & \in S^n, \quad b_i \in \mathbb{R}^n, \quad c_i \in \mathbb{R}, \quad i = 0, 1, \ldots, m.
\end{align*}
\]

Homogeneous IQCQP (HIQCQP):

\[
\begin{align*}
\text{Min} & \quad x^T A_0 x \\
\text{s.t.} & \quad x^T A_i x + c_i \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]
SCOP for Convex QCQP

The popularity of SOCP is also due to the fact that it is a generalized form of convex QCQP (Quadratically Constrained Quadratic Programming). Specifically, consider the following QCQP:

\[
\begin{align*}
\text{Min} & \quad x^T A_0 x + 2b_0^T x + c_0 \\
\text{s.t.} & \quad x^T A_i x + 2b_i^T x + c_i \leq 0, \quad i = 1, \ldots, m
\end{align*}
\]

where \( A_0 \succeq 0, \ A_i \succeq 0 \) for \( i = 1, \ldots, m \).

Note that

\[
t \geq \sum_{i=1}^n x_i^2 \iff \left\| \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ (t-1)/2 \end{bmatrix} \right\|_2 ^2 \leq \frac{t+1}{2} \iff \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ (t-1)/2 \\ (t+1)/2 \end{bmatrix} \in \mathcal{L}^{n+2}
\]
SOCP for Convex QCQP

Therefore, for each $i = 1, \ldots, m$

$$x^T A_i x + 2b_i^T x + c_i \leq 0 \iff \begin{bmatrix} A_i^{1/2} x \\ -1/2 - b_i^T x - c_i/2 \\ 1/2 - b_i^T x - c_i/2 \end{bmatrix} \in \mathcal{L}^{n+2}$$

QCQP can be equivalently written as

$$\begin{array}{ll}
\text{Min} & u \\
\text{s.t.} & \begin{bmatrix}
A_0^{1/2} x \\
-1/2 - b_0^T x + u/2 - c_0/2 \\
1/2 - b_0^T x + u/2 - c_0/2 \\
A_i^{1/2} x \\
-1/2 - b_i^T x - c_i/2 \\
1/2 - b_i^T x - c_i/2
\end{bmatrix} \in \mathcal{L}^{n+2}, \\
i = 1, \ldots, m.
\end{array}$$
Nonconvex QCQP and SDP

SDP Relaxation for Nonconvex QCQP

Recall that

\[ x^T A_i x + 2b_i^T x + c_i = \begin{bmatrix} 1 \end{bmatrix}^T \begin{bmatrix} c_i & b_i^T \\ b_i & A_i \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} c_i \\ b_i \\ A_i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ x \\ x^T x^T \end{bmatrix} \]

Let \( Y = \begin{bmatrix} 1 \\ x \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^T x \end{bmatrix} = \begin{bmatrix} 1 \\ x^T x \\ x x^T \end{bmatrix} \in S^{n+1}_+, \) we have SDP Relaxation

\[
\begin{align*}
\text{Min} & \quad \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} \cdot Y \\
\text{s.t.} & \quad \begin{bmatrix} c_i & b_i^T \\ b_i & A_i \end{bmatrix} \cdot Y \leq 0, \quad i = 1, \ldots, m \\
& \quad Y_{11} = 1 \\
& \quad Y \succeq 0, \quad \text{rank}(Y) = 1
\end{align*}
\]

Note: No convexity is assumed.
SDP Solution

Exact Global Solution from SDP Relaxation

Theorem
When $m = 1$, if the optimal solution of the SDP relaxation for QCQP exists, then there exists a rank one optimal solution $Y^*$ of the SDP relaxation for QCQP, and, furthermore, let $Y^* = \begin{bmatrix} 1 & (x^*)^T \\ x^* & x^*(x^*)^T \end{bmatrix}$, then $x^*$ is optimal for QCQP.
SDP Solution

Exact Global Solution from SDP Relaxation

**Theorem**

When $m = 2$, assume there exists $\bar{x}$ such that $\bar{x}^T A_i \bar{x} + 2 b_i^T \bar{x} + c_i < 0$, $i = 1, 2$, and there exists $\bar{y}_1 \geq 0$, $\bar{y}_2 \geq 0$ such that

$$\begin{pmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{pmatrix} + \bar{y}_1 \begin{pmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{pmatrix} + \bar{y}_2 \begin{pmatrix} c_2 & b_2^T \\ b_2 & A_2 \end{pmatrix} > 0.$$ 

If, for the optimal solution $Y^*$, either $\begin{pmatrix} c_1 & b_1^T \\ b_1 & A_1 \end{pmatrix} \cdot Y^* < 0$ or $\begin{pmatrix} c_2 & b_2^T \\ b_2 & A_2 \end{pmatrix} \cdot Y^* < 0$, then there is no gap between QCQP and its SDP relaxation, and at least one optimal solution can be obtained from the rank-one decomposition of $Y^*$. 
Extended QCQP

Extension of QCQP

Ext-QCQP

\[
\begin{align*}
\text{Min} & \quad x^T A_0 x + 2b_0^T x + c_0 \\
\text{s.t.} & \quad x^T A_i x + 2b_i^T x + c_i \leq 0, \quad i = 1, \ldots, m \\
& \quad (x_1^2, \ldots, x_n^2)^T \in \mathcal{X}
\end{align*}
\]  

(Ext-QCQP)

where \( \mathcal{X} \) is a closed convex set.

SDP Relaxation:

\[
\begin{align*}
\text{Min} & \quad \begin{bmatrix} c_0 & b_0^T \\ b_0 & A_0 \end{bmatrix} \cdot Y \\
\text{s.t.} & \quad \begin{bmatrix} c_i & b_i^T \\ b_i & A_i \end{bmatrix} \cdot Y \leq 0, \quad i = 1, \ldots, m \\
& \quad (Y_{22}, Y_{33}, \ldots, Y_{(n+1),(n+1)})^T \in \mathcal{X} \\
& \quad Y_{11} = 1 \\
& \quad Y \succeq 0, \quad \text{rank}(Y) = 1
\end{align*}
\]
Extended QCQP

Exact Solution from SDP Relaxation

Theorem
Suppose there is $\sigma \in \{-1, 1\}^{n+1}$, such that all the off-diagonal elements of $\Lambda_{\sigma} \begin{bmatrix} c_i & b_i^T \\ b_i & A_i \end{bmatrix} \Lambda_{\sigma}$, $0 \leq i \leq m$, are nonpositive. Here

$$(\Lambda_{\sigma})_{ij} = \begin{cases} 
\sigma_i & i = j \\
0 & i \neq j
\end{cases}$$

If $Y^*$ is an optimal solution for the SDP relaxation for Ext-QCQP, then

$$x^* = \sigma_1(\sigma_2\sqrt{Y_{22}^*}, \sigma_3\sqrt{Y_{33}^*}, \ldots, \sigma_{n+1}\sqrt{Y_{(n+1),(n+1)}})^T$$

is optimal for Ext-QCQP.
More about QCQP and LCoP?

• The key lies in constructing a suitable cone for quadratic functions.

• This leads to the concepts of “Cone of Non-negative Quadratic Functions” and “Cone of Non-negative Quadratic Forms.”
Cone of Nonnegative Quadratic Functions

- Nonnegative quadratic functions over $\mathcal{F} \subset \mathbb{R}^n$

$$f(x) = x^T A x + 2 b^T x + c \geq 0, \forall x \in \mathcal{F}$$

$$f \Leftrightarrow \begin{bmatrix} c & b^T \\ b & A \end{bmatrix}$$

- $\mathcal{D}_\mathcal{F} = \{ \begin{bmatrix} c & b^T \\ b & A \end{bmatrix} \in S^{n+1} | \begin{bmatrix} 1^T \\ x \end{bmatrix} \begin{bmatrix} c & b^T \\ b & A \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0, \forall x \in \mathcal{F} \}$ is a closed, convex cone.

- $\mathcal{D}_\mathcal{F}^* = \text{cl}(\text{cone}\{ \begin{bmatrix} 1 & x^T \\ x & xx^T \end{bmatrix} | x \in \mathcal{F} \})$

- $(\mathcal{D}_\mathcal{F}^*)^* = \mathcal{D}_\mathcal{F}$ and $(\mathcal{D}_\mathcal{F})^* = \mathcal{D}_\mathcal{F}^*$
Cone of Nonnegative Quadratic Functions

- Examples:
  - $\mathcal{F} = \mathbb{R}^n$
    - $\mathcal{D}_\mathcal{F} = \mathcal{D}^*_\mathcal{F} = S_{+}^{n+1}$
  - $\mathcal{F} = \mathbb{R}_+^n$
    - $\mathcal{D}_\mathcal{F} = C_{n+1}$ and $\mathcal{D}^*_\mathcal{F} = C^*_{n+1}$
Connection between QP and LCoP

\[
\begin{align*}
\min \quad & x^T Q x + 2b^T x + c \\
\text{s.t.} \quad & x \in \mathcal{F}.
\end{align*}
\tag{QP(\mathcal{F})}
\]

\[
\inf \begin{bmatrix} c & b^T \\ b & Q \end{bmatrix} \cdot X \quad \sup \quad \sigma
\]
\[s.t. \quad X_{11} = 1, \quad \text{(LCoP)} \quad s.t. \quad \begin{bmatrix} c - \sigma & b^T \\ b & Q \end{bmatrix} \in \mathcal{D}_\mathcal{F}, \quad \text{(LCoD)}\]
\[
X \in \mathcal{D}_\mathcal{F}^*, \quad \sigma \in \mathbb{R}.
\]

- \(\mathcal{D}_\mathcal{F}^* = \text{cl cone}\left\{ X = \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^T \in S^{n+1} \mid x \in \mathcal{F} \right\} \)
- \(\mathcal{D}_\mathcal{F} = \left\{ M \in S^{n+1} \mid \begin{bmatrix} 1 \\ x \end{bmatrix}^T M \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0, \ \forall x \in \mathcal{F} \right\} \)
Connection between QP and LCoP

- $\mathcal{D}_F$ and $\mathcal{D}_F^*$ are both closed and convex. They are dual to each other.

- Sturm and Zhang (2003) proved that if $\mathcal{F}$ is not empty, then $\text{QP}(\mathcal{F})$, $\text{LCoP}(\mathcal{F})$ and $\text{LCoD}(\mathcal{F})$ have the same optimal objective value.

Sufficient Condition for Global Optimality

Lemma (Positive semidefiniteness condition).
Let \((x^*, \lambda^*)\) be a KKT solution. If \(A(\lambda^*) \in S^n_+\), then \(x^*\) is a globally optimal solution of QCQP.

Theorem (Copositiveness condition)
Let \((x^*, \lambda^*)\) be a KKT solution. If \(M(x^*, \lambda^*) \in D_{n+1}\), then \(x^*\) is an optimal solution of the QCQP problem, where

\[
M(x^*, \lambda^*) = \begin{bmatrix}
c(\lambda^*) - f(x^*) & b^T(\lambda^*) \\
b(\lambda^*) & A(\lambda^*)
\end{bmatrix}.
\]
Finding the Lagrangian Multiplier Vector $\lambda^*$

Assumption
There exists a KKT solution $(x^*, \lambda^*)$ satisfying the LICQ condition and copositiveness condition.

Lemma
Under the copositiveness condition assumption, $(\sigma^*, \lambda^*) = (f(x^*), \lambda^*)$ is an optimal solution of CoD1. Furthermore, if the LICQ assumption holds, then for any $(\bar{\sigma}, \bar{\lambda})$ being an optimal solution of the CoD1 problem, we have $\bar{\lambda} \leq_{\mathbb{R}^n_{+}} \lambda^*$. 
Finding the Lagrangian Multiplier Vector $\lambda^*$

Theorem
Let $(x^*, \lambda^*)$ be a KKT solution. If it satisfies the LICQ condition and the copositiveness condition, then $\lambda^*$ is the unique optimal solution of the maximization problem CoD2

$$\max \quad \sum_{i=1}^{m} \lambda_i$$

s.t.  
$$\begin{bmatrix} c(\lambda) - \sigma^* \\ b(\lambda) \\ b^T(\lambda) \\ A(\lambda) \end{bmatrix} \in \mathcal{D}_{n+1}, \quad (\text{CoD2})$$

$$\lambda \succeq_{\mathbb{R}^n_+} 0,$$

where $\sigma^*$ is the optimal objective value obtained by solving CoD1. Furthermore, if $A(\lambda^*)$ is invertible, then

$$x^* = -A^{-1}(\lambda^*)b(\lambda^*).$$
Relaxing the Conic Constraint

Choose a computable cone $\mathcal{K}_{n+1}$ such that $S_+^{n+1} \subset \mathcal{K}_{n+1} \subset \mathcal{D}_{n+1}$.

CoD1':

\[
\begin{align*}
\text{Max} & \quad \sigma \\
\text{s.t.} & \quad \begin{bmatrix} c(\lambda) - \sigma & b^T(\lambda) \\ b(\lambda) & A(\lambda) \end{bmatrix} \in \mathcal{K}_{n+1}, \\
& \quad \lambda \geq_{\mathbb{R}^n_+} 0.
\end{align*}
\]

CoD2':

\[
\begin{align*}
\text{Max} & \quad \sum_{i=1}^m \lambda_i \\
\text{s.t.} & \quad \begin{bmatrix} c(\lambda) - \sigma^* & b^T(\lambda) \\ b(\lambda) & A(\lambda) \end{bmatrix} \in \mathcal{K}_{n+1}, \\
& \quad \lambda \geq_{\mathbb{R}^n_+} 0.
\end{align*}
\]
Algorithm (QCQP)

**STEP 1:** Given a QCQP problem, solve the corresponding CoD1’ problem and get the optimal value $\sigma^*$. If failed, then stop and the problem cannot be solved by the current scheme.

**STEP 2:** Solve CoD2’ to get the optimal $\lambda^*$.

**STEP 3:** Compute $x^* = -A^+(\lambda^*) b(\lambda^*)$.

**STEP 4:** If $(x^*, \lambda^*)$ is a KKT solution, then return $x^*$ as a global optimal solution of QCQP with the objective value $f(x^*) = \sigma^*$. Otherwise, return $\sigma^*$ as a lower bound of QCQP.

Here $A^+$ is the Moore-Penrose generalized inverse for a given square matrix $A \in \mathbb{R}^{n \times n}$. 