# □ Duality Theory

$$\begin{array}{lll} & \text{Min} & \mathbf{c}^T\mathbf{x} \\ \text{(LP)} & \text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \quad \text{Primal Problem} \\ & & \mathbf{x} \geq 0 \end{array}$$

Assume:

$\mathbf{x}$ is a nondegenerate bfs.

$$\mathbf{A} = [\ \mathbf{B} \mid \mathbf{N}\ ]$$

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_B \\ \hline \mathbf{x}_N \end{bmatrix} \quad \mathbf{c}^T = [\ \mathbf{c}_B^T \mid \mathbf{c}_N^T\ ]$$

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{B}^{-1} & -\mathbf{B}^{-1}\mathbf{N} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

$$\mathbf{d}_q = \begin{bmatrix} -\mathbf{B}^{-1}\mathbf{A}_q \\ \hline e_q \end{bmatrix}, x_q : \text{n.b.v.}$$

$$
\begin{aligned}
\mathbf{c}^T \mathbf{d}_q &= [\ \mathbf{c}_B^T \mid \mathbf{c}_N^T\ ] \left[ \begin{array}{c} -\mathbf{B}^{-1}\mathbf{A}_q \\ \hline e_q \end{array} \right] \\
&= -\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A}_q + c_q \\
&= r_q
\end{aligned}
$$

<u>Fact:</u>

$\mathbf{x}$ is optimal $\iff r_q \geq 0 \ \ \forall\, q \in \mathbf{N}$.

*i.e.*, $\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A}_q \leq c_q \ \ \forall\, q \in \mathbf{N}$

also $\mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{A}_p = c_p \ \ \forall\, p \in \mathbf{B}$

Define $\mathbf{c}_B^T \mathbf{B}^{-1} := \mathbf{w}^T \in R^m$

$\mathbf{w} = (w_1, w_2, \ldots, w_m)^T$ simplex multiplier

Hence $\mathbf{w}^T[\ \mathbf{B} \mid \mathbf{N}\ ] \leq \mathbf{c}^T$

or equivalently $\mathbf{A}^T\mathbf{w} \leq \mathbf{c}$

Moreover,

$$\begin{aligned} \mathbf{b}^T\mathbf{w} &= \mathbf{w}^T\mathbf{b} \\ &= \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} \\ &= \mathbf{c}_B^T\mathbf{x}_B \\ &= \mathbf{c}^T\mathbf{x} \quad (\mathbf{x}_N = 0) \end{aligned}$$

But, in general, for $\mathbf{A}\bar{\mathbf{x}} = \mathbf{b},\ \bar{\mathbf{x}} \geq 0$

$\mathbf{b}^T\mathbf{w} = \mathbf{w}^T\mathbf{b} = \mathbf{w}^T\mathbf{A}\bar{\mathbf{x}} \leq \mathbf{c}^T\bar{\mathbf{x}}$

Hence we define an associated problem

$$\begin{array}{lll} & \text{Max} & \mathbf{b}^T\mathbf{w} \\ \text{(D)} & \text{s. t.} & \mathbf{A}^T\mathbf{w} \leq \mathbf{c} \quad \text{Dual Problem} \\ & & \mathbf{w} \in R^m \end{array}$$

## Observation:

(1) Problem (D) is a linear program with $m$ variables and $n$ constraints. The right-hand-side vector and the cost vector change roles in (P) and (D).

(2) Both (P) and (D) are defined by the same data set $(\mathbf{A}, \mathbf{b}, \mathbf{c})$.

(3) What's the linear dual problem of the dual problem?

## Standard Form $\mathbf{w} = \mathbf{u} - \mathbf{v}$

$$-\text{Min} \quad [-\mathbf{b}^T \mid \mathbf{b}^T \mid 0\,] \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{s} \end{bmatrix}$$

$$\text{(P)} \qquad \text{s. t.} \quad [\,\mathbf{A}^T \mid -\mathbf{A}^T \mid I\,] \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{s} \end{bmatrix} = \mathbf{c}$$

$$\mathbf{u}, \mathbf{v}, \mathbf{s} \geq 0$$

$$
(D) \qquad \begin{array}{ll} -\text{Max} & \mathbf{c}^T\mathbf{w} \\ \text{s. t.} & \begin{bmatrix} \mathbf{A} \\ -\mathbf{A} \\ I \end{bmatrix} \mathbf{w} \leq \begin{bmatrix} -\mathbf{b} \\ \mathbf{b} \\ 0 \end{bmatrix} \\ & \mathbf{w} \text{ unrestricted} \in R^n \end{array}
$$

or

$$
\begin{array}{rl} -\text{Max} & \mathbf{c}^T\mathbf{w} \\ \text{s. t.} & \mathbf{A}\mathbf{w} \leq -\mathbf{b} \\ & -\mathbf{A}\mathbf{w} \leq \mathbf{b} \\ & \mathbf{w} \leq 0 \end{array}
$$

or

$$
\begin{array}{rl} \text{Min} & -\mathbf{c}^T\mathbf{w} \\ \text{s. t.} & -\mathbf{A}\mathbf{w} = \mathbf{b} \\ & \mathbf{w} \leq 0 \end{array}
$$

For $\mathbf{x} := -\mathbf{w}$, we have

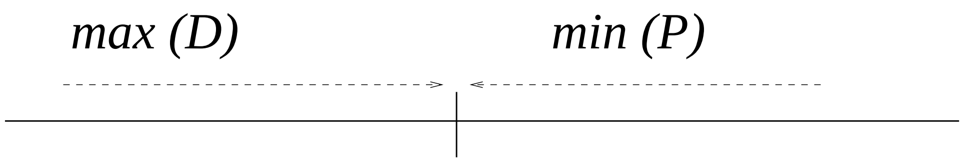
$$
\begin{array}{rl} \text{Min} & \mathbf{c}^T\mathbf{x} \\ \text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}
$$

**Lemma 4.1** Dual of the Dual=Primal.

(4) If $\mathbf{x}$ is primal feasible, $\mathbf{w}$ is dual feasible, then

$$\begin{aligned} \mathbf{c}^T\mathbf{x} &= \mathbf{x}^T\mathbf{c} \\ &\geq \mathbf{x}^T\mathbf{A}^T\mathbf{w} \\ &= \mathbf{b}^T\mathbf{w} \end{aligned}$$

( Weak Duality Theorem )




(5) If $\mathbf{x}$ is primal feasible, $\mathbf{w}$ is dual feasible, and $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{w}$,
then $\mathbf{x}$ is primal optimal, and $\mathbf{w}$ is dual optimal.

(6) If the primal is unbounded below, then the dual is infeasible.

(7) If the dual is unbounded above, then the primal is infeasible.

(8) Is the converse statement of (6) or (7) true?
No! may be both infeasible.

(9) We can derive a stronger result:

**Theorem 4.2** (Strong Duality Theorem)

(a) If either the primal or the dual has a finite optimum, then so does the other and $\min \mathbf{c}^T\mathbf{x} = \max \mathbf{b}^T\mathbf{w}$ ( No duality gap! )

(b) If either problem has an unbounded objective, then the other has no feasible solution.

Proof:

(a) Because of observations (3) & (5), we only have to show that "*if the primal has a (finite) optimal bfs* $\mathbf{x}$*, the* $\exists$ *a dual feasible solution* $\mathbf{w}$ *such that* $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{w}$. "
We apply the Simplex method at $\mathbf{x}$, let $\mathbf{w} = (\mathbf{B}^T)^{-1}\mathbf{c}_B$ be the simplex multiplier,

then

$$
\begin{aligned}
\mathbf{c} - \mathbf{A}^T\mathbf{w} &= \begin{bmatrix} \mathbf{c}_B \\ \mathbf{c}_N \end{bmatrix} - \begin{bmatrix} \mathbf{B}^T \\ \mathbf{N}^T \end{bmatrix} \mathbf{w} \\
&= \begin{bmatrix} \mathbf{c}_B - \mathbf{B}^T(\mathbf{B}^T)^{-1}\mathbf{c}_B \\ \mathbf{c}_N - \mathbf{N}^T(\mathbf{B}^T)^{-1}\mathbf{c}_B \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ r_N \end{bmatrix} \geq 0
\end{aligned}
$$

Thus $\mathbf{w}$ is dual feasible. Moreover,

$$
\begin{aligned}
\mathbf{c}^T\mathbf{x} &= \mathbf{c}_B^T\mathbf{x}_B \\
&= \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} \\
&= \mathbf{w}^T\mathbf{b} \\
&= \mathbf{b}^T\mathbf{w}.
\end{aligned}
$$

(b) Duality from the Weak Duality Theorem.

## Implications:

(1) The simplex multiplier $\mathbf{w}$ corresponding to a primal optimal solution $\mathbf{x}$ is a dual optimal solution.

(2) At each iteration of the simplex method, the simplex multiplier $\mathbf{w}$ always satisfies that $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{w}$.

However, $\mathbf{w}$ is not dual feasible unless $r_N \geq 0$.

(3) Revised Simplex Method

Keep primal feasibility
and $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{w}$ (no duality gap)
but seeks for dual feasibility.

## Applications:

### (1) Theorem of Alternatives

"Existence of solutions of systems of equalities and inequalities"

**Theorem 4.3** (Farkas Lemma)

The system

$$\text{(I)} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq 0$$

has no solution if and only if the system

$$\text{(II)} \quad \mathbf{A}^T\mathbf{w} \leq 0, \quad \mathbf{b}^T\mathbf{w} > 0$$

has solution.

Proof: Consider LP

$$\begin{array}{lll} \text{(P)} & \text{Min} & 0^T\mathbf{x} \\ & \text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & & \mathbf{x} \geq 0 \end{array}$$

and its dual

$$\begin{array}{lll} \text{(D)} & \text{Max} & \mathbf{b}^T\mathbf{w} \\ & \text{s. t.} & \mathbf{A}^T\mathbf{w} \leq 0 \end{array}$$

Since $\mathbf{w} = 0$ is dual feasible , we know

(P) is infeasible $\Leftrightarrow$ (D) is unbounded. Now,

(P) is infeasible $\Leftrightarrow$ (I) has no solution.

(D) is unbounded $\Leftrightarrow$ (II) has a solution. Hence,

(I) has no solution $\Leftrightarrow$ (II) has a solution.

<u>Rewording</u> Farkas Lemma

Two systems

$$\begin{array}{lll} \text{(I)} & \mathbf{A}\mathbf{x} = \mathbf{b}, & \mathbf{x} \geq 0 \\ \text{(II)} & \mathbf{A}^T\mathbf{w} \leq 0, & \mathbf{b}^T\mathbf{w} > 0 \end{array}$$

Either (I) or (II) has a solution but NOT both.

## (2) Complementary Slackness

*symmetric pair*

$$
\begin{array}{lll}
\text{(P)} & \min & \mathbf{c}^T\mathbf{x} \\
& \text{s. t.} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\
& & \mathbf{x} \geq 0
\end{array}
$$

$$
\begin{array}{ll}
\min & \mathbf{c}^T\mathbf{x} + 0^T\mathbf{s} \\
\text{s. t.} & \mathbf{A}\mathbf{x} - I\mathbf{s} = \mathbf{b} \\
& \mathbf{x}, \mathbf{s} \geq 0
\end{array}
$$

$$
\begin{array}{ll}
\max & \mathbf{b}^T\mathbf{w} \\
\text{s. t.} & \begin{bmatrix} \mathbf{A}^T \\ -I \end{bmatrix} \mathbf{w} \leq \begin{bmatrix} \mathbf{c} \\ \mathbf{0} \end{bmatrix}
\end{array}
$$

$$
\begin{array}{lll}
\text{(D)} & \max & \mathbf{b}^T\mathbf{w} \\
& \text{s. t.} & \mathbf{A}^T\mathbf{w} \leq \mathbf{c} \\
& & \mathbf{w} \geq 0
\end{array}
$$

Let $\mathbf{x}$ be primal feasible, $\mathbf{w}$ be dual feasible.
Define

$$\begin{aligned} \mathbf{s} &= \mathbf{A}\mathbf{x} - \mathbf{b} && \geq 0 \\ \mathbf{r} &= \mathbf{c} - \mathbf{A}^T\mathbf{w} && \geq 0 \end{aligned}$$

$\mathbf{s} \in R^m$ : <u>primal slackness</u>
$\mathbf{r} \in R^n$ : <u>dual slackness</u>

## <u>Observation 1:</u>

$$\left\{ \begin{array}{ll} \text{If} & \mathbf{r}^T\mathbf{x} = 0 \\ \text{and} & \mathbf{w}^T\mathbf{s} = 0 \end{array} \right\} \text{complementary slackness}$$

then

$$\underline{(\mathbf{c}^T - \mathbf{w}^T\mathbf{A})}\mathbf{x} = \mathbf{0}$$

and

$$\mathbf{w}^T\underline{(\mathbf{A}\mathbf{x} - \mathbf{b})} = \mathbf{0}$$

Hence
$\mathbf{c}^T\mathbf{x} = \mathbf{w}^T\mathbf{A}\mathbf{x} = \mathbf{w}^T\mathbf{b} = \mathbf{b}^T\mathbf{w}$

Thus $\mathbf{x}$ is primal optimal and $\mathbf{w}$ is dual optimal.

## Observation 2:

On the contrary side, for a feasible pair $(\mathbf{x}, \mathbf{w})$,

$$\mathbf{c}^T\mathbf{x} \geq \mathbf{w}^T\mathbf{A}\mathbf{x} \geq \mathbf{w}^T\mathbf{b}$$

If $\mathbf{x}$ is primal optimal and $\mathbf{w}$ is dual optimal, then $\mathbf{c}^T\mathbf{x} = \mathbf{w}^T\mathbf{A}\mathbf{x} = \mathbf{w}^T\mathbf{b}$

Hence

$$(\mathbf{c}^T - \mathbf{w}^T\mathbf{A})\mathbf{x} = \mathbf{0}$$

and

$$\mathbf{w}^T(\mathbf{A}\mathbf{x} - \mathbf{b}) = \mathbf{0}$$

In summary, we have

**Theorem 4.4** (Complementary Slackness)

Let (P) and (D) be a "symmetric pair",
$\mathbf{x}$ is primal feasible, $\mathbf{w}$ is dual feasible.
Then $\mathbf{x}$, $\mathbf{w}$ are optimal solution pair if and only if

$$\begin{cases} r_j = 0 \text{ or } x_j = 0 \quad \forall\ j = 1, 2, \ldots, n. \\ s_i = 0 \text{ or } w_i = 0 \quad \forall\ i = 1, 2, \ldots, m. \end{cases}$$

Special Case:

$$\text{(P)} \quad \min \quad \mathbf{c}^T\mathbf{x} \qquad \text{s. t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq 0$$

$$\text{(D)} \quad \max \quad \mathbf{b}^T\mathbf{w} \qquad \text{s. t.} \quad \mathbf{A}^T\mathbf{w} \leq \mathbf{c}$$

the condition $\mathbf{w}^T\mathbf{s} = \mathbf{0}$ is always true.
The complementary slackness condition reduces to $\mathbf{r}^T\mathbf{x} = \mathbf{0}$.

**Theorem 4.5** ( Kuhn-Tucker Condition )
$\mathbf{x}$ is optimal for the problem

$$\begin{aligned} \text{(P)} \quad \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{s. t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

if and only if there exist $\mathbf{w}$ and $\mathbf{r}$ such that

(1)$\mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0$ (Primal feasibilty)

(2)$\mathbf{A}^T\mathbf{w} + \mathbf{r} = \mathbf{c},\ \mathbf{r} \geq 0$ (Dual feasibilty)

(3)$\mathbf{r}^T\mathbf{x} = \mathbf{0}$ (Complementary Slackness)

Proof: Direct Consequence of Thm 4.4.

# Economic Interpretation of Duality:

## (1) Dual Variables

$$
\begin{array}{llll}
\text{(P)} & \min & \mathbf{c}^T\mathbf{x} & \text{(minimize total cost)} \\
 & \text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} & \text{(satisfy demands)} \\
 & & \mathbf{x} \geq 0 & \text{(different services)}
\end{array}
$$

Assume that $\mathbf{x}^*$ nondegenerate optimal bfs

$$\mathbf{x}^* = \begin{pmatrix} \mathbf{x}_B^* \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ 0 \end{pmatrix}$$

$$z^* = \mathbf{c}^T\mathbf{x}^* = \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}$$

Since $\mathbf{x}_B^* = \mathbf{B}^{-1}\mathbf{b} > 0$
Thus $\mathbf{B}^{-1}(\mathbf{b} + \triangle\mathbf{b}) > 0$ when $\triangle\mathbf{b}$ is small enough! then

$$\bar{\mathbf{x}}^* = \begin{pmatrix} \bar{\mathbf{x}}_B^* \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}(\mathbf{b} + \triangle\mathbf{b}) \\ 0 \end{pmatrix}$$

is an optimal bfs to

$$
\begin{aligned}
(\bar{\text{P}}) \quad \min \quad & \mathbf{c}^T\mathbf{x} \\
\text{s. t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} + \triangle\mathbf{b} \\
& \mathbf{x} \geq 0
\end{aligned}
$$

with $\bar{z}^* = \mathbf{c}_B^T\mathbf{B}^{-1}(\mathbf{b} + \triangle\mathbf{b})$
(Why? No change in $r_q$! )
Moreover,

$$
\begin{aligned}
\triangle z \quad & = \bar{z}^* - z^* \\
& = \mathbf{c}_B^T\mathbf{B}^{-1}(\mathbf{b} + \triangle\mathbf{b}) - c_B^T B^{-1}\mathbf{b} \\
& = \underbrace{\mathbf{c}_B^T\mathbf{B}^{-1}}_{\mathbf{w}^T} \triangle \mathbf{b}
\end{aligned}
$$

(simplex multiplier for (P) at optimum!)

Hence, $w_i$: " marginal price" of the $i$th demand.
Note:
$w_i$ indicates the minimum unit price that one has to charge for additional demand $i$.
It is also called " shadow price" or " equilibrium price".

## (2) Dual Problem

Scenario:

$n$ products to produce

$$x_j = \text{amount of product } j, \;\; j = 1, 2, \ldots, n$$

$m$ resources in hand

$$b_i = \text{amount of resource } i, \;\; i = 1, 2, \ldots, m$$

Selling Prices

$$c_1, c_2, \ldots, c_n$$

Technology matrix

$[\ a_{ij}\ ]$ : each product $j$ consumes $a_{ij}$ units of resource $i$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

# A manufacturer's point of view:

- Maximize total sales (hence profit)

$$\sum_{j=1}^{n} c_j x_j$$

- Resource limitation

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq b_i, \ i = 1, 2, \ldots, m$$

- Production Requirement
  $x_j \geq 0, \ j = 1, 2, \ldots, n$

$$\begin{array}{lll} \text{(P)} & \max & \mathbf{c}^T\mathbf{x} \\ & \text{s. t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & & \mathbf{x} \geq 0 \end{array}$$

$$\begin{array}{lll} \text{(D)} & \min & \mathbf{b}^T\mathbf{w} \\ & \text{s. t.} & \mathbf{A}^T\mathbf{w} \geq \mathbf{c} \\ & & \mathbf{w} \geq 0 \end{array}$$

## Getting Resources

- $m$ resources to purchase from a supplier
  $w_i$= unit price to purchase resource $i$,
  $i = 1, 2, \ldots, m$

- Free information market:
  Supplier knows your selling price $c_j$ for product $x_j$ and he/she wants to get the most out of you. *i.e.*

  $$a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{mj}w_m \geq c_j, \; j = 1, 2, \ldots, n$$

  Therefore, we have the dual

$$\begin{array}{lll} \min & \mathbf{b}^T\mathbf{w} & \text{(minimize total spending)} \\ \text{s. t.} & \mathbf{A}^T\mathbf{w} \geq \mathbf{c} & \text{(price accepted by the supplier)} \\ & \mathbf{w} \geq 0 & \end{array}$$

## Observations

(i) $w_i^*$ is the max marginal price the manufacturer is willing to pay the supplier for resource $i$.

(ii) When resource $i$ is not fully utilized, *i.e.*

$$a_{i1}x_1^* + a_{i2}x_2^* + \ldots + a_{in}x_n^* < b_i$$

the complementary slackness condition
$\Rightarrow w_i^* = 0$.
This means the manufacturer is not willing to pay a penny for any additional amount!

(iii) When the supplier asks too much, *i.e.*

$$a_{1j}w_1 + a_{2j}w_2 + \ldots + a_{mj}w_n > c_j$$

then $x_j = 0$.
This means the manufacturer is not going to produce any product $j$!

# Sensitivity Analysis

- Post-optimality analysis

$$\begin{aligned} \min \quad & \mathbf{c}^T \mathbf{x} \\ \text{s. t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

| | | |
|---|---|---|
| $\mathbf{c}$ : | may change | (cost) |
| $\mathbf{b}$ : | may change | (resource) |
| $\mathbf{A}$ : | may change | (technology matrix) |

Question:

Will $\mathbf{x}^*$ remain optimal ?

$\mathbf{B}^*$ remain optimal ?

or How will they change accordingly?

Case 1 ($\mathbf{c}$ changes)

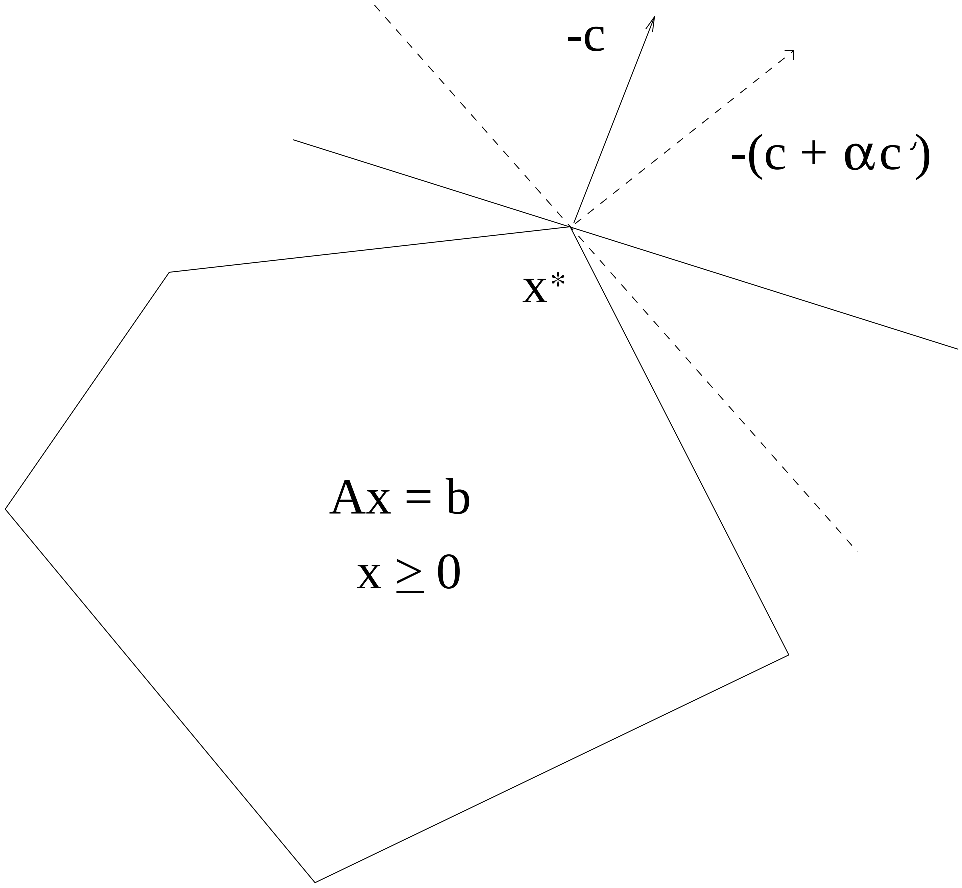
$\mathbf{x}^*$ optimal solution, $\mathbf{A} = [\mathbf{B}|\mathbf{N}]$

Given $\mathbf{c}' = \left[\begin{smallmatrix} \mathbf{c}'_B \\ \mathbf{c}'_N \end{smallmatrix}\right] \in R^n$ be a perturbation,

$$\mathbf{c} \longrightarrow \mathbf{c} + \alpha\mathbf{c}' = \begin{bmatrix} \mathbf{c}_B + \alpha\mathbf{c}'_B \\ \mathbf{c}_N + \alpha\mathbf{c}'_N \end{bmatrix} \stackrel{\triangle}{=} \bar{\mathbf{c}},$$

$$\alpha \in R$$

$$\begin{array}{lll} & \min & z(\alpha) = \underline{(\mathbf{c} + \alpha\mathbf{c}')^T}\mathbf{x} = \bar{\mathbf{c}}^T\mathbf{x} \\ (\mathrm{P}') & \text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & & \mathbf{x} \geq 0 \end{array}$$




Fact: As $\alpha$ changes,
$\mathbf{x}^*(\alpha)$ changes, $\mathbf{B}(\alpha)$ changes,
$z^*(\alpha)$ changes

Question: Within which range $[\ \underline{\alpha},\ \bar{\alpha}\ ]$,
$\mathbf{x}^*$ remain optimal?

## Analysis:

(1) (P) and (P′) have the same feasible domain, hence $\mathbf{x}^*$ is feasible to (P′) for any $\alpha$.

(2) $\mathbf{x}^*$ remains optimal to (P′) if

$$\bar{r}_N^T = \bar{\mathbf{c}}_N^T - \bar{\mathbf{c}}_B^T \mathbf{B}^{-1}\mathbf{N} \geq 0$$

$i.e.,$

$$(\mathbf{c}_N + \alpha \mathbf{c'}_N)^T - (\mathbf{c}_B + \alpha \mathbf{c'}_B)^T \mathbf{B}^{-1}\mathbf{N} \geq 0$$

$$\underbrace{(\mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{N})}_{r_N^T} + \alpha \underbrace{(\mathbf{c'}_N^T - \mathbf{c'}_B^T \mathbf{B}^{-1}\mathbf{N})}_{r'^T_N} \geq 0$$

$i.e.,$

$$\alpha r'^T_N \geq -r_N^T$$

(3) Case 1. For $r_q^{'} > 0,\ q \in N$

$\alpha \geq \frac{-r_q}{r_q^{'}}$ is required, thus

$$\underline{\alpha} = \max\{\frac{-r_q}{r_q^{'}} \mid r_q^{'} > 0,\ q \in \mathbf{N}\}$$

otherwise

$$\underline{\alpha} = -\infty,\ \text{if } r_q^{'} \leq 0,\ \forall\ q \in \mathbf{N}$$

(4) Case 2. For $r_q' < 0, q \in \mathbf{N}$

$\alpha \leq \frac{-r_q}{r_q'}$ is required, thus

$$\bar{\alpha} = \min\{\frac{-r_q}{r_q^{'}} \mid r_q^{'} < 0,\ q \in \mathbf{N}\}$$

otherwise

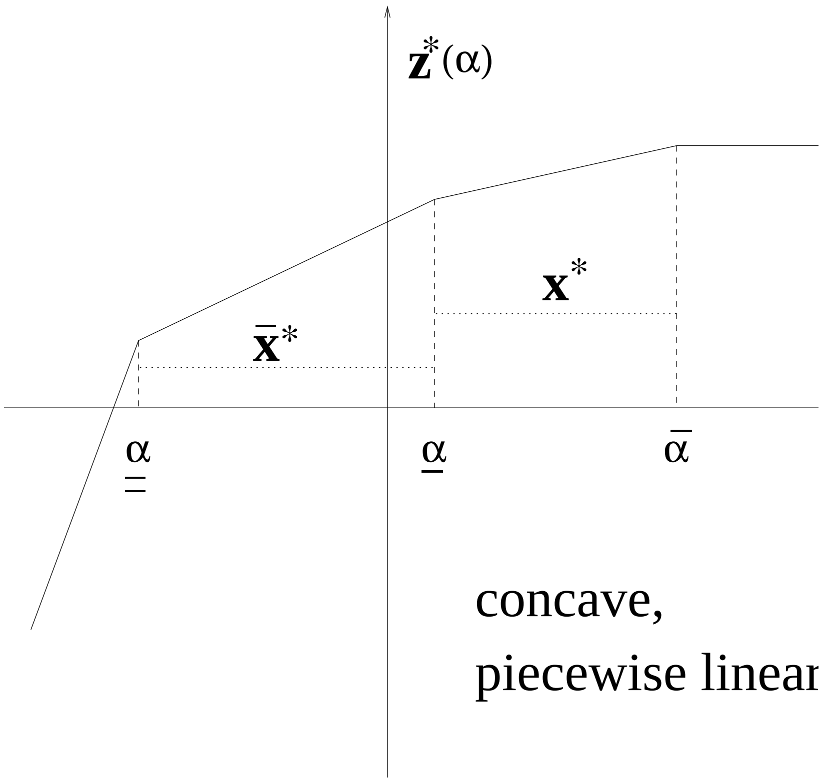
$$\bar{\alpha} = \infty,\ \text{if } r_q^{'} \geq 0,\ \forall\ q \in \mathbf{N}$$

(5) For $\alpha \in [\ \underline{\alpha},\ \bar{\alpha}\ ]$, $\mathbf{x}^*$ remains optimal.

$$
\begin{aligned}
z^*(\alpha) &= (\mathbf{c}_B^T + \alpha \mathbf{c'}_B^T)\mathbf{B}^{-1}\mathbf{b} \\
&= \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b} + \alpha \underbrace{\mathbf{c'}_B^T\mathbf{B}^{-1}\mathbf{b}}_{k:\ \text{constant}} \\
&= z^* + k\ \alpha
\end{aligned}
$$

Thus $z^*(\alpha)$ is linear in $\alpha$.

(6) .



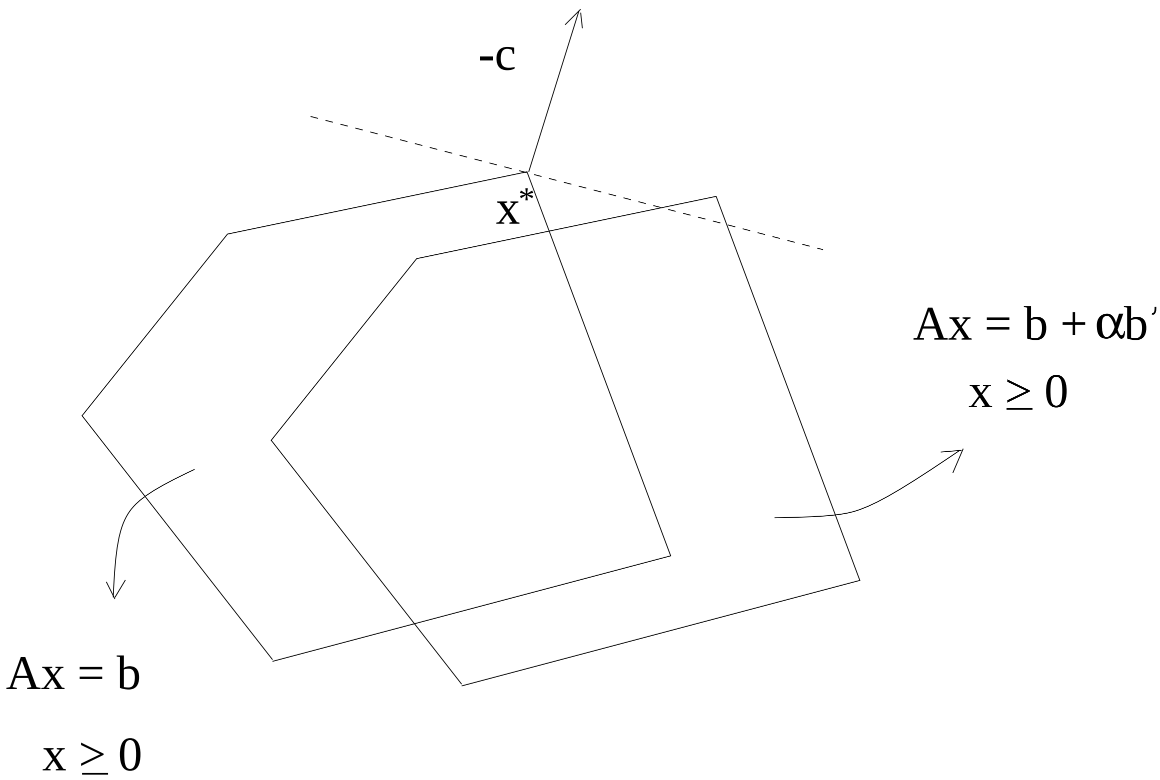

(7) Take

$$\mathbf{c}' = e_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \rightarrow jth\ place$$

then $[\ c_j - \underline{\alpha},\ c_j + \bar{\alpha}\ ]$ gives stable range of the $j$th cost coefficient, or how sensitive the cost component is.

Case 2 (**b** changes)
Let $\mathbf{b}' \in R^m$ be a perturbation.

$$
(P') \quad \begin{array}{ll} \min & \mathbf{c}^T \mathbf{x} \\ \text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} + \alpha \mathbf{b}' \\ & \mathbf{x} \geq 0 \end{array}
$$




Note: $\mathbf{x}^*$ may become infeasible !

<u>Question:</u> Within which range $[\ \underline{\alpha},\ \bar{\alpha}\ ]$, will $\mathbf{B}$ remain as an optimal basis?

<u>Analysis:</u>

(1) $\mathbf{B}$ is an optimal basis if

(i) ${r_N}^T = \mathbf{c}_N^T - \mathbf{c}_B^T \mathbf{B}^{-1}\mathbf{N} \geq 0$, and

(ii) $x(\alpha) = \left[ \frac{\mathbf{B}^{-1}(\mathbf{b}+\alpha\mathbf{b}')}{\mathbf{0}} \right] \geq 0$

(2) (i) always holds, since $\mathbf{c}, \mathbf{B}, \mathbf{N}$ no change!

but (ii) is not always true.

(3) We need $\mathbf{B}^{-1}(\mathbf{b} + \alpha\mathbf{b}') \geq 0$

*i.e.*, $\underbrace{\mathbf{B}^{-1}\mathbf{b}}_{\bar{\mathbf{b}}} + \alpha \underbrace{\mathbf{B}^{-1}\mathbf{b}'}_{\bar{\mathbf{b}}'} \geq 0$

*i.e.*, $\alpha\bar{\mathbf{b}}' \geq -\bar{\mathbf{b}}$

(i) For $\bar{\mathbf{b}'}_p > 0,\ p \in \mathbf{B}$, we need

$$\alpha \geq \frac{-\bar{\mathbf{b}}_p}{\bar{\mathbf{b}'}_p}$$

$$\text{Thus } \underline{\alpha} = \begin{cases} \max\{\frac{-\bar{b}_p}{\bar{b}'_p} \mid \bar{b}'_p > 0, p \in \mathbf{B}\}, \\ -\infty \end{cases}$$

(ii) For $\bar{\mathbf{b}'}_p < 0,\ p \in \mathbf{B}$, we need

$$\alpha \leq \frac{-\bar{\mathbf{b}}_p}{\bar{\mathbf{b}'}_p}$$

$$\text{Thus } \bar{\alpha} = \begin{cases} \min\{\frac{-\bar{b}_p}{\bar{b}'_p} \mid \bar{b}'_p < 0,\ p \in \mathbf{B}\}, \\ +\infty \end{cases}$$

(4) When $\alpha \in [\ \underline{\alpha},\ \bar{\alpha}\ ]$,

$$\mathbf{x}^*(\alpha) = \left( \frac{\mathbf{B}^{-1}(\mathbf{b}+\alpha\mathbf{b}')}{\mathbf{0}} \right)$$

$$= \left( \frac{\mathbf{B}^{-1}\mathbf{b}+\alpha\mathbf{B}^{-1}\mathbf{b}'}{\mathbf{0}} \right)$$

$$= \left( \frac{\mathbf{B}^{-1}\mathbf{b}}{\mathbf{0}} \right) + \alpha \left( \frac{\mathbf{B}^{-1}\mathbf{b}'}{\mathbf{0}} \right)$$

$$= \mathbf{x}^* + \alpha \left( \frac{\mathbf{B}^{-1}\mathbf{b}'}{\mathbf{0}} \right)$$

linear in $\alpha$!

(5)

$$\begin{aligned} z^*(\alpha) &= \mathbf{c}_B^T \mathbf{x}^*(\alpha)_B \\ &= \mathbf{c}_B^T(\mathbf{x}_B^* + \alpha\mathbf{B}^{-1}\mathbf{b}') \\ &= \mathbf{c}_B^T\mathbf{x}_B^* + \alpha\underbrace{\mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{b}'}_{k} \\ &= z^* + k\ \alpha \end{aligned}$$

again, linear in $\alpha$!

Case 3 (**A** changes):
Usually not so simple.

## (Special Case i)

### Adding a new variable

*A new product, activity becomes available.*

$$(P') \quad \begin{array}{ll} \min & \mathbf{c}^T\mathbf{x} + c_{n+1}x_{n+1} \\ \text{s. t.} & \mathbf{A}\mathbf{x} + A_{n+1}x_{n+1} = \mathbf{b} \\ & \mathbf{x}, x_{n+1} \geq 0 \end{array}$$

## Analysis:

(1) $\left[\frac{\mathbf{x}^*}{0}\right]$ is a bfs of (P′) with $[\ \mathbf{B} \mid \mathbf{N}, A_{n+1}]$.

(2) $\left[\frac{\mathbf{x}^*}{0}\right]$ is an optimal solution of (P′) if

$$r_{n+1} = c_{n+1} - \mathbf{c}_B^T \mathbf{B}^{-1} A_{n+1} \geq 0.$$

(3) If $r_{n+1} < 0$ , then $x_{n+1}$ enters the basis and continue the revised simplex method to find an optimal solution of (P′).

## (Special Case ii)

### Removing a variable

*An activity $x_k$ is no longer available.*

(a) if $x_k^* = 0$, then $\mathbf{x}^*$ remains optimal by deleting $x_k^*$.

(b) if $x_k^* > 0$, then $\mathbf{x}_k$ has to leave the basis. Can this be done ? Consider

$$\text{(Phase I)} \quad \begin{array}{ll} \min & x_k \\ \text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

(1) $\mathbf{x}^*$ is a current bfs to start the revised simplex solution.

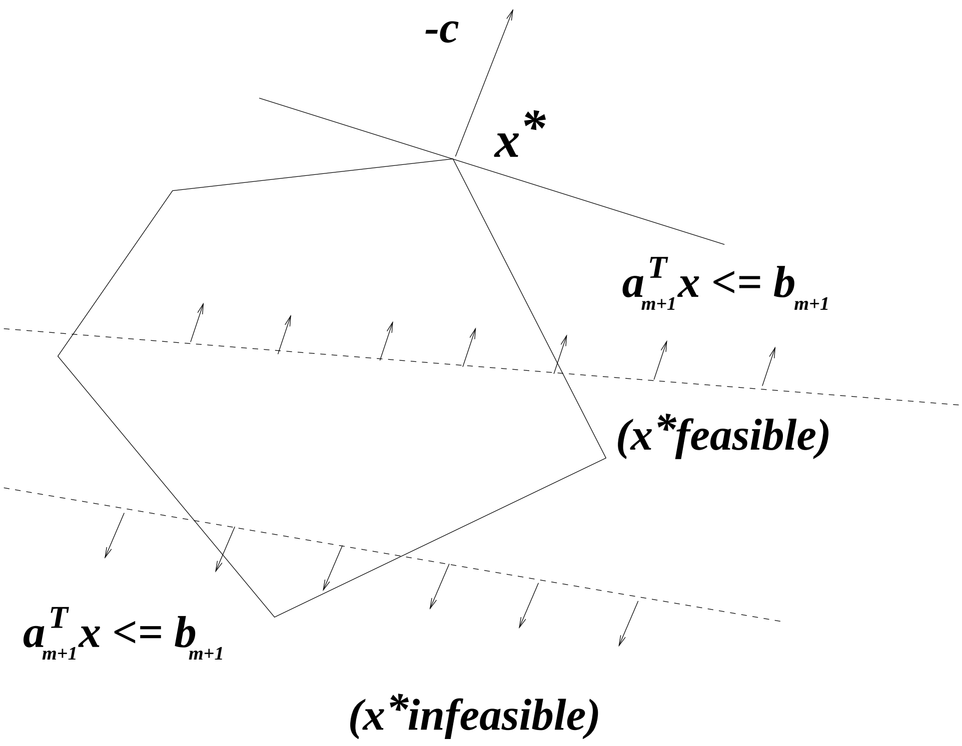
(2) If $z_{PhI}^* > 0$, then removing $x_k$ will cause infeasibility.
If $z_{PhI}^* = 0$, then we can start from there to solve the new problem.

## (Special Case iii)

### Adding a new constraint

$$
\begin{array}{lll}
 & \min & \mathbf{c}^T\mathbf{x} \\
(P') & \text{s. t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\
 & & \mathbf{a}_{m+1}^T\mathbf{x} \leq b_{m+1} \\
 & & \mathbf{x} \geq 0
\end{array}
$$

$$\mathbf{a}_{m+1}^T = (a_{m+1,1}, a_{m+1,2}, \ldots, a_{m+1,n})$$



Analysis:

(1) If $\mathbf{a}_{m+1}^T \mathbf{x}^* \le b_{m+1}$ then $\mathbf{x}^*$ remains optimal!

(2) If not, $\mathbf{x}^*$ is not feasible and we have to find a new basis of dimensionality $m+1$.

(3) Consider

$$
\begin{array}{lll}
 & \min & \mathbf{c}_B^T \mathbf{x}_B + \mathbf{c}_N^T \mathbf{x}_N + 0 x_{n+1} \\
(P') & \text{s. t.} & \mathbf{B}\mathbf{x}_B + \mathbf{N}\mathbf{x}_N = \mathbf{b} \\
 & & (\mathbf{a}_{m+1})_B^T \mathbf{x}_B + (\mathbf{a}_{m+1})_N^T \mathbf{x}_N + x_{n+1} = b_{m+1} \\
 & & \mathbf{x}_B, \mathbf{x}_N, x_{n+1} \ge 0
\end{array}
$$

$$
\bar{\mathbf{B}} := \begin{pmatrix} \mathbf{B} & \mathbf{0} \\ (\mathbf{a}_{m+1})_B^T & 1 \end{pmatrix}
$$

then $\bar{\mathbf{B}}$ is a nonsingular $(m+1) \times (m+1)$ matrix, and

$$
\bar{\mathbf{B}}^{-1} = \begin{pmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ -(\mathbf{a}_{m+1})_B^T \mathbf{B}^{-1} & 1 \end{pmatrix}
$$

*i.e.* $\bar{\mathbf{B}}$ is a basis for (P′)!

(4) The reduced cost

$$
\begin{aligned}
r'_q &= c_q - [\ \tfrac{\mathbf{c}_B}{0}\ ]^T \bar{\mathbf{B}}^{-1} [\tfrac{A_q}{a_{m+1,q}}] \\
&= c_q - [\ \mathbf{c}_B^T \mathbf{B}^{-1} \mid 0\ ][\tfrac{A_q}{a_{m+1,q}}] \\
&= c_q - \mathbf{c}_B^T \mathbf{B}^{-1} A_q \\
&= r_q,\ \forall q \in \mathbf{N}
\end{aligned}
$$

Since $\mathbf{B}$ is an optimal basis to (P), we know
$r'_q = r_q \geq 0,\ \ \forall q \in \mathbf{N}$
$i.e$, $\bar{\mathbf{B}}$ provides a dual feasible solution
$\mathbf{w}^T = \mathbf{c}_{\bar{B}}^T \bar{\mathbf{B}}^{-1}$ for (P′).

(5) Define

$$
\begin{aligned}
\bar{\mathbf{x}}_{\bar{B}} &= \bar{\mathbf{B}}^{-1}(\tfrac{\mathbf{b}}{b_{m+1}}) \\
\bar{\mathbf{x}}_N &= \mathbf{0}
\end{aligned}
$$

then $\bar{\mathbf{x}} = (\frac{\bar{\mathbf{x}}_{\bar{B}}}{\bar{\mathbf{x}}_N})$ is an optimal solution of (P′) if $\bar{\mathbf{x}}_{\bar{B}} \geq 0$.

(6) If

$$
\bar{\mathbf{x}}_{\bar{B}} = \bar{\mathbf{B}}^{-1}(\frac{\mathbf{b}}{b_{m+1}}) \ngeq 0
$$

then we can apply the dual simplex method with

$$\begin{aligned}
\mathbf{w}^T &= \mathbf{c}_{\bar{B}}^T \bar{\mathbf{B}}^{-1} \\
&= (\mathbf{c}_B^T,\ 0) \begin{pmatrix} \mathbf{B}^{-1} & \mathbf{0} \\ -(\mathbf{a}_{m+1})_B^T \mathbf{B}^{-1} & 1 \end{pmatrix} \\
&= (\mathbf{c}_B^T \mathbf{B}^{-1} \mid 0)
\end{aligned}$$

to solve (P′).

## Solving the Dual

$$\text{(D)} \qquad \max \quad \mathbf{b}^T\mathbf{w} \qquad m \text{ variables}$$
$$\text{s. t.} \quad \mathbf{A}^T\mathbf{w} \le \mathbf{c} \qquad n \text{ constraints}$$

$$\text{(D}'\text{)} \quad (-)\min \quad -\mathbf{b}^T\mathbf{u} + \mathbf{b}^T\mathbf{v} + 0^T\mathbf{s} \qquad 2m+n \text{ variables}$$
$$\text{s. t.} \quad \mathbf{A}^T\mathbf{u} - \mathbf{A}^T\mathbf{v} + \mathbf{s} = \mathbf{c} \qquad n \text{ constraints}$$
$$\mathbf{u}, \mathbf{v}, \mathbf{s} \ge 0$$

## Apply Revised Simplex Method to (D′)

(i) Dimensionality becomes larger.

(ii) Keep (*dual*) feasibility
Maintain complementary slackness
Seek (*primal*) feasibility

# Dual Simplex Method

- Basic Idea

(1) Start with a basis

$$\mathbf{A} = [\ \mathbf{B} \mid \mathbf{N}\ ]$$

such that
$\mathbf{w}^T := \mathbf{c}_B^T \mathbf{B}^{-1}$ is dual feasible, *i.e.*,
$\mathbf{A}^T \mathbf{w} \leq \mathbf{c}$

(2) Further define

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{b} \\ 0 \end{pmatrix}$$

then

$$\begin{aligned} \mathbf{r}^T \mathbf{x} &= (\mathbf{c}^T - \mathbf{w}^T \mathbf{A})\mathbf{x} \\ &= \mathbf{c}^T \mathbf{x} - \mathbf{w}^T \mathbf{A} \mathbf{x} \\ &= \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b} - \mathbf{c}_B^T \mathbf{B}^{-1} \mathbf{b} \\ &= 0 \end{aligned}$$

*i.e.*, complementary slackness condition holds.

(3) Since

$$\mathbf{A}\mathbf{x} = [\ \mathbf{B} \mid \mathbf{N}\ ][\ \frac{\mathbf{x}_B}{\mathbf{x}_N}\ ] = \mathbf{B}\mathbf{B}^{-1}\mathbf{b} = \mathbf{b}.$$

If $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b} \geq 0$, then
we have primal feasibility
and $\mathbf{x} = [\ \frac{\mathbf{x}_B}{\mathbf{x}_N}\ ]$ is primal optimal,
$\mathbf{w}^T := \mathbf{c}_B^T\mathbf{B}^{-1}$ is dual optimal.

If there exists $p \in \mathbf{B}$ such that $x_p < 0$
then $A_p$ may leave the basis
and $x_p \longleftarrow 0$ becomes nbv.
And we have to pivot-in a nbv $x_q$ for $q \in \mathbf{N}$.

## Question 1:

Which $x_q$ is entering the basis?

## Analysis:

(1) We should keep dual feasibility and complementary slackness.

(2) Complementary slackness condition always holds according to the way we define $\mathbf{w}$ and $\mathbf{x}$.

(3) The real problem is to keep dual feasibility while $x_p$ leaves and $x_q$ enters.

Question 2:

Where is the dual feasibility information?

Guess: Fundamental martix

$$\mathbf{M}^{-1} = \begin{pmatrix} \mathbf{B}^{-1} & -\mathbf{B}^{-1}\mathbf{N} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

$$\begin{aligned} \mathbf{c}^T\mathbf{M}^{-1} &= (\ \mathbf{c}_B^T \mid \mathbf{c}_N^T\ ) \begin{pmatrix} \mathbf{B}^{-1} & -\mathbf{B}^{-1}\mathbf{N} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \\ &= (\ \mathbf{c}_B^T\mathbf{B}^{-1} \mid \mathbf{c}_N^T - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{N}\ ) \\ &= (\ \mathbf{w}^T \mid \mathbf{r}_N^T\ ) \end{aligned}$$

$\nearrow$ dual variable $\qquad$ $\nwarrow$ reduced cost

$$r_N \geq 0 \iff \text{dual feasibility}$$

Answer:

$$x_p \longrightarrow \text{nonbasic}$$

$$x_q \longleftarrow \text{basic}$$

$$\mathbf{M} \longleftarrow \text{update}$$

We check

$$\mathbf{c}^T\mathbf{M}^{-1} = (\ \mathbf{w}^T \mid \mathbf{r}_N^T\ )$$

for dual feasibility!

Question 3:

$$x_p \longrightarrow \text{nonbasic}$$

$$x_q \longleftarrow \text{basic}$$

How will $\mathbf{M}^{-1}$ change?

Answer: Sherman-Morrison-Woodbury formula.

## Lemma 4.2

$M$ : nonsingular $n \times n$ matrix.

$$u = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in R^n$$

If $w := 1 + v^T M^{-1} u \neq 0$

then

$(M + uv^T)$ is nonsingular and

$$(M + uv^T)^{-1} = M^{-1} - (1/w) M^{-1} uv^T M^{-1}.$$

Proof: Check that

$(M + uv^T)[M^{-1} - (1/w) M^{-1} uv^T M^{-1}] = I.$

## Observations:

(1) It takes $o(n^3)$ elementary operations to invert the matrix $(M + uv^T)$ directly.

(2) The formula only takes $o(n^2)$ elementary operations.

(3) Sometimes, we call it *rank-one* updating method.

(4) For our case

$$\left\{ \begin{array}{c} x_p \longrightarrow \text{out} \\ x_q \longleftarrow \text{in} \end{array} \right\} M \text{ becomes } \bar{M}$$

$$\bar{M} = M + e_q(e_p - e_q)^T.$$

## Example:

$$A = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \left[\begin{array}{cc|ccc} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \end{array}\right] \end{array}$$

$$M = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \longrightarrow e_5^T$$

$x_1 \to$ out, $p = 1$, $x_5 \to$ in, $q = 5$

$$\bar{M} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \longrightarrow e_1^T$$

The $q$th row ($e_q^T$) of $M$ is replaced by $e_p^T$ in $\bar{M}$, hence

$\bar{M} = M + e_q(e_p - e_q)^T =$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \end{pmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 & 9 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \bar{M}.
\end{aligned}
$$

Let $u = e_q,\ v = (e_p - e_q)$

$$
\bar{M}^{-1} \quad = M^{-1} - \frac{M^{-1} + e_q(e_p - e_q)^T M^{-1}}{1 + (e_p - e_q)^T M^{-1} e_q}
$$

Notice that

$$
\begin{aligned}
e_q^T M^{-1} \quad &= q\text{th row of } M^{-1} \\
&= e_q^T
\end{aligned}
$$

Hence

$$
\begin{aligned}
\bar{M}^{-1} \quad &= M^{-1} - \frac{M^{-1} e_q (e_p^T M^{-1} - e_q^T)}{1 + e_p^T M^{-1} e_q - e_q^T e_q} \\
&= M^{-1} - \frac{M^{-1} e_q (e_p^T M^{-1} - e_q^T)}{e_p^T M^{-1} e_q}
\end{aligned}
$$

Since $(\mathbf{w}, r_N^T) = c^T M^{-1}$, we have

$$\begin{aligned}
(\bar{\mathbf{w}}, \bar{r}_N^T) &= c^T \bar{M}^{-1} \\
&= c^T (M^{-1} - \tfrac{M^{-1} e_q (e_p^T M^{-1} - e_q^T)}{e_p^T M^{-1} e_q}) \\
&= (\mathbf{w}, r_N^T) - \tfrac{c^T M^{-1} e_q (e_p^T M^{-1} - e_q^T)}{e_p^T M^{-1} e_q}
\end{aligned}$$

We define

$$\begin{aligned}
u^T &= e_p^T \mathbf{B}^{-1} \\
y_j &= u^T A_j \\
\gamma &= \tfrac{r_q}{y_q}
\end{aligned}$$

then

$$\begin{aligned}
\bar{\mathbf{w}} &= \mathbf{w} + \gamma u \\
\bar{r}_j &= r_j - \gamma y_j, \ \forall j \in \mathbf{N} - \{q\} \\
\bar{r}_p &= -\gamma
\end{aligned}$$

## Observatios:

1. $u^T = e_p^T \mathbf{B}^{-1} \Longrightarrow u^T$ is the $p$th row of $\mathbf{B}^{-1}$.
2. $y_q = u^T A_q = -d_p^q$ where $\mathbf{d}^q = -\mathbf{B}^{-1} A_q$.
3. To maintain dual feasibility, we need
$\bar{r}_p = -\gamma = -\frac{r_q}{y_q} \geq 0$
and
$\bar{r}_j = r_j - \gamma y_j \geq 0, \ \forall j \in \mathbf{N} - \{q\}$.

## Case 1:

If $\exists j \in \mathbf{N}$ such that $y_j < 0$
then $\frac{-r_j}{y_j} \geq -\gamma$ is required.
Therefore $q$ is chosen by the min-ratio test, *i.e.*,

$$\frac{-r_q}{y_q} = \min\{\frac{-r_j}{y_j} \mid y_j < 0, j \in \mathbf{N}\}$$

## Case 2:

$$\begin{array}{rl} \text{If} & y_j \geq 0,\ \forall j \in \mathbf{N} \\ \text{then} & y_j = u^T A_j = e_p^T \mathbf{B}^{-1} A_j,\ j \in \mathbf{N}. \\ \text{and} & e_p^T \mathbf{B}^{-1} \mathbf{A} = e_p^T \mathbf{B}^{-1} [\ \mathbf{B} \mid \mathbf{N}\ ] \geq 0. \\ \text{Thus} & e_p^T \mathbf{B}^{-1} \mathbf{A} \mathbf{x} \geq 0,\ \forall \mathbf{x} \geq 0,\ \mathbf{x} \text{ feasible.} \\ & \| \\ & e_p^T \mathbf{B}^{-1} \mathbf{b} = e_p^T \mathbf{x}_B = x_p.\ \ \rightarrow\leftarrow x_p < 0 \end{array}$$

Therefore there is no primal feasible solution!

## Step-by-Step Algorithm

§ 5.3

# How to start the Dual Simplex Method?

$$(P) \quad \min \quad \mathbf{c}^T\mathbf{x} \quad \text{s. t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b}, \quad \mathbf{x} \geq 0$$

$$(D) \quad \max \quad \mathbf{b}^T\mathbf{w} \quad \text{s. t.} \quad \mathbf{A}^T\mathbf{w} \leq \mathbf{c}$$

$\mathbf{A} = [\ \mathbf{B} \mid \mathbf{N}\ ]$, $\mathbf{B}_{m\times m}$ nonsingular matrix.
$\mathbf{w}^T = \mathbf{c}_B^T\mathbf{B}^{-1}$
$\mathbf{A}^T\mathbf{w} \leq \mathbf{c}$?

## If not, consider:

$$(P') \quad \min \quad \mathbf{c}^T\mathbf{x} \quad \text{s. t.} \quad \mathbf{A}\mathbf{x} = \mathbf{B}e, \quad \mathbf{x} \geq 0$$

$$(D') \quad \max \quad e^T\mathbf{B}^T\mathbf{w} \quad \text{s. t.} \quad \mathbf{A}^T\mathbf{w} \leq \mathbf{c}$$

$e^T = (1, 1, \ldots, 1)$

Observations:

(1) $\mathbf{x} = \binom{e}{0}$ is a primal bfs of (P′).

(2) (D) and (D′) have the same feasible domain.

(3) Apply the revised simplex method to (P′), either it stops at an optimal solution, or find (P′) is unbounded.

(4) If it stops at an optimal solution, then $\mathbf{w}^{*T} = \mathbf{c}_{B^*}^T(\mathbf{B}^*)^{-1}$ is feasible to (D′). Hence $\mathbf{w}^*$ is feasible to (D).

(5) If (P′) is unbounded, then we find a feasible direction $\mathbf{d}$, such that $\mathbf{Ad} = 0,\ \mathbf{d} > 0$ and $\mathbf{c}^T\mathbf{d} < 0$.
Hence (P) is also unbounded
and (D) must be infeasible!

Remarks:

(1) Solving a standard form LP by the dual simplex method is mathematically equivalently to solving its dual LP by the revised (primal) simplex method.

(2) The work of solving an LP by the dual simplex method is about the same as of by the revised (primal) simplex method.

(3) The dual simplex method is useful for the sensitivity analysis.

# □ Computer Implementation of the Dual Simplex Method

**Step 1** (starting with a feasible basic solution):

In the primal problem, given,

$\mathbf{B} = [\mathbf{A}_{j_1}, \mathbf{A}_{j_2}, \mathbf{A}_{j_3}, \ldots \mathbf{A}_{j_m}]$

$\widetilde{\mathbf{B}} = [j_1, j_2, j_3, \ldots j_m]$

A dual basic feasible solution $\mathbf{w}$ can be obtained by solving

$$\mathbf{B}^{\mathbf{T}}\mathbf{w} = \mathbf{c}_B$$

Compute the reduced cost $\mathbf{r}$ with

$$r_j = c_j - \mathbf{w}^{\mathbf{T}}\mathbf{A}_j, \quad \forall j \notin \widetilde{\mathbf{B}}$$

**Step 2** (checking for optimality):

Compute $\mathbf{x}_B$ by solving

$$\mathbf{B}\mathbf{x}_B = \mathbf{b}$$

If $\mathbf{x}_B \geq \mathbf{0}$, then STOP. The current solution

$$\left[\begin{array}{c}\mathbf{x}_B \\ \hline \mathbf{x}_N\end{array}\right] = \left[\begin{array}{c}\mathbf{B}^{-1}\mathbf{b} \\ \hline \mathbf{0}\end{array}\right]$$

is optimal.

Otherwise go to Step 3.

**Step 3** (leaving the basis):

Choose a basic variable $x_{j_p} < 0$ with index $j_p \in \widetilde{\mathbf{B}}$.

**Step 4** (checking for infeasibility):

Compute $\mathbf{u}$ by solving

$$\mathbf{B}^T \mathbf{u} = \mathbf{e}_p$$

Also compute

$$y_j = \mathbf{u}^T \mathbf{A}_j, \quad \forall j \notin \widetilde{\mathbf{B}}$$

If $y_j \geq 0, \ \forall j \notin \widetilde{\mathbf{B}}$; then STOP. The primal problem is infeasible.

Otherwise go to Step 5.

**Step 5** (entering the basis):

Choose a nonbasic variable $x_q$ by the minimum ratio test

$$\frac{-r_q}{y_q} = \min \left\{ \frac{-r_j}{y_j} \;\middle|\; y_j < 0, j \notin \widetilde{\mathbf{B}} \right\}.$$

Set

$$\frac{-r_q}{y_q} = -\gamma$$

**Step 6** (updating the reduced costs):

$$\begin{aligned} r_j &\leftarrow r_j - \gamma y_j \quad \forall j \notin \widetilde{\mathbf{B}}, \quad j \neq q \\ r_{j_p} &\leftarrow -\gamma \end{aligned}$$

**Step 7** (updating the current solution and basis):

Compute $\mathbf{d}$ by solving

$$\mathbf{Bd} = -\mathbf{A}_q$$

Set

$$\begin{aligned} x_q &\leftarrow \alpha = \frac{x_{j_p}}{y_q} = \left(\frac{-x_{j_p}}{d_p}\right) \\ x_{j_i} &\leftarrow x_{j_i} + \alpha d_{j_i}, \qquad \forall j_i \in \widetilde{\mathbf{B}}, \quad i \neq p \\ \mathbf{B} &\leftarrow \mathbf{B} + [A_q - A_{j_p}]\mathbf{e}_p^T \\ \widetilde{\mathbf{B}} &\leftarrow \widetilde{\mathbf{B}} \cup \{q\} \setminus \{j_p\} \end{aligned}$$

Go to Step 2.

**Example**

$$\begin{array}{lrrrrcr} \text{Minimize} & -2x_1 & -x_2 & & & & \\ \text{s.t} & x_1 & +x_2 & +x_3 & & = & 2 \\ & x_1 & +x_4 & & & = & 1 \\ & x_1, & x_2, & x_3, & x_4 & \geq & 0 \end{array}$$

**Step 1** (starting): Choose $\widetilde{\mathbf{B}} = \{1, 4\}$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{c}_B = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Then the dual solution

$$\mathbf{w}^{\mathbf{T}} = \mathbf{c}_B^T \mathbf{B}^{-1} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Computing $r_j, \ \forall j \notin \widetilde{\mathbf{B}}$, we have $r_2 = 1, r_3 = 2$, which implies that $\mathbf{w}$ is dual feasible.

**Step 2** (checking for optimality):

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

the corresponding primal vector is infeasible.

**Step 3** (leaving the basis):

$x_4 < 0$ (the second element in $\widetilde{\mathbf{B}}$), $x_4$ leaves the basis and let $p = 2$.

**Step 4** (checking infeasibility):

$$\mathbf{u}^T = \mathbf{e}_2^T \mathbf{B}^{-1} = [-1 \quad 1]$$

and

$$y_2 = \mathbf{u}^T \mathbf{A}_2 = -1, \quad y_3 = \mathbf{u}^T \mathbf{A}_3 = -1$$

**Step 5** (entering the basis):

Take the minimum ratio test

$$-\frac{r_2}{y_2} = \min\left\{ \frac{-1}{-1}, \ \frac{-2}{-1} \right\} = 1 = -\gamma$$

Therefore $x_2$ is entering the basis and $p = 2$.

**Step 6** (updating the reduced cost):

$$r_4 = -\gamma = 1 \text{ and } r_3 = 2 - \gamma y_3 = 1$$

(note that $r_2$ has been changed from 1 to 0 as

$x_2$ enters the basis.)

**Step 7** (updating current solution and basis):
Solving for $\mathbf{d}$ in $\mathbf{Bd} = -\mathbf{A}_2$, we obtain

$$\mathbf{d} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Also

$$x_2 = \alpha = \frac{x_4}{y_2} = 1$$

$$x_1 = 2 - 1 \times 1 = 1$$

Thus the new primal vector has $x_1 = x_2 = 1$ (and nonbasic variables $x_3 = x_4 = 0$).
Since it's nonnegative, we know it's a optimal solution to the original linear program.
The corresponding optimal basis $\mathbf{B}$ becomes

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$