□ **Geometry of LP**

- Terminologies:

$$\text{(LP)} \qquad \begin{aligned} &\text{Min } \mathbf{c}^T\mathbf{x} \\ &\text{s. t. } \mathbf{A}\mathbf{x} = \mathbf{b} \\ &\mathbf{x} \geq 0 \end{aligned}$$

(1) $P = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$
feasible domain

(2) When $P \neq \phi$, LP is *consistent.*
$\mathbf{x}$ is a *feasible solution* if $\mathbf{x} \in P$.

(3) P is *bounded* if

$$\exists\ M > 0 \text{ such that } \|\mathbf{x}\| \leq M,\ \forall\ \mathbf{x} \in P.$$

In this case, LP has a *bounded feasible domain*.

(4) LP is *bounded* if

$$\exists\ M \in R \text{ such that } \mathbf{c}^T\mathbf{x} \geq M\ \forall\ \mathbf{x} \in P.$$

LP has a bounded feasible domain.

(5) $\Downarrow\ \Uparrow ?$

LP is bounded.

(6) $\mathbf{x}^*$ is an *optimal solution* if

$$\mathbf{x}^* \in P \text{ and } \mathbf{c}^T\mathbf{x}^* = \underset{x \in P}{\text{Min}}\, \mathbf{c}^T\mathbf{x}$$

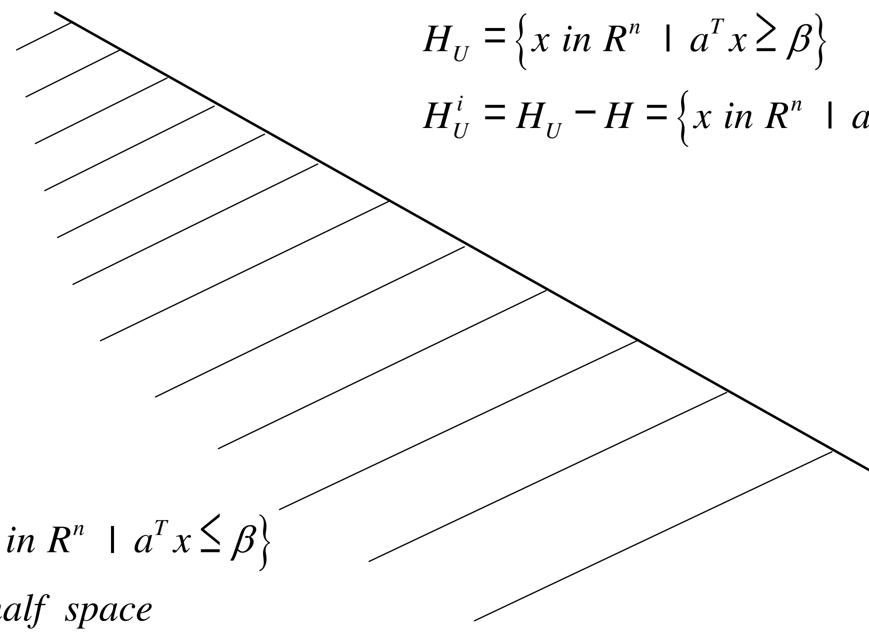
$P^* = \{\mathbf{x}^* \mid \mathbf{x}^* \text{ is optimal}\}$ *optimal solution set*.

$\mathbf{x}^*$ *solves* LP, if $\mathbf{x}^* \in P^*$.

- <u>Definition:</u>

For a vector $\mathbf{a} \in \mathbf{R}^n, \mathbf{a} \neq 0$, and a scalar $\beta \in \mathbf{R}$, define

$$H = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{a}^T\mathbf{x} = \beta\} \ \underline{\textit{hyperplane}}$$



$$H_U = \{x \text{ in } R^n \mid a^T x \geq \beta\}$$

$$H_U^i = H_U - H = \{x \text{ in } R^n \mid a^T x > \beta\}$$

$$H_L = \{x \text{ in } R^n \mid a^T x \leq \beta\}$$

closed half space

$$H_L^i = H_L - H = \{x \text{ in } R^n \mid a^T x < \beta\}$$

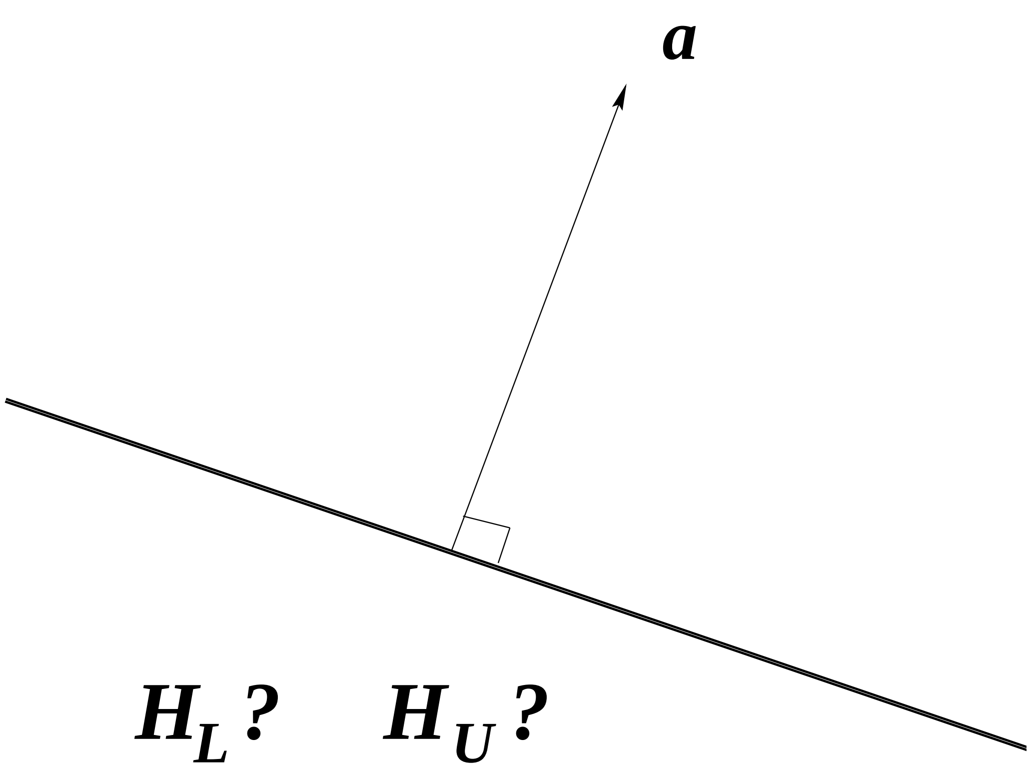
open half space

H is the <u>*bounding hyperplane*</u> of H_L and H_L^i.
$\mathbf{a}$ is the <u>*normal*</u> of H.

Property 1: $\mathbf{a}$ is orthogonal to all vectors parallel to H.

Proof:

$$\begin{aligned} &\forall\ \mathbf{y}, \mathbf{z} \in H, \\ \mathbf{a}^T(\mathbf{y}-\mathbf{z}) &= \mathbf{a}^T\mathbf{y} - \mathbf{a}^T\mathbf{z} \\ &= \beta - \beta = 0. \end{aligned}$$

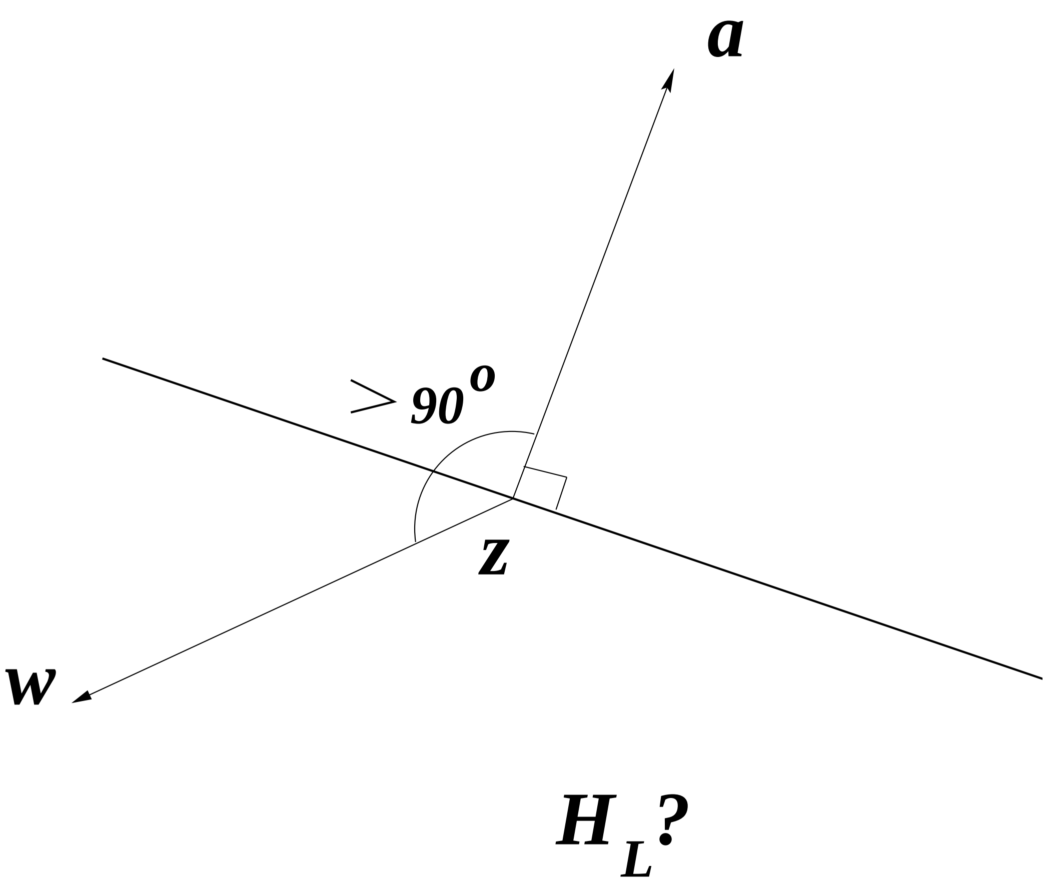


Property 2: $\mathbf{a}$ is directed toward H_U.

Proof:

For any $\mathbf{z} \in H, \mathbf{w} \in H_L^i$,

$$\begin{aligned} \mathbf{a}^T(\mathbf{w} - \mathbf{z}) &= \mathbf{a}^T\mathbf{w} - \mathbf{a}^T\mathbf{z} \\ &< \beta - \beta = 0. \end{aligned}$$



- Definition:

A *polyhedral set* or *polyhedron* is a set formed by the intersection of a finite number of a closed half spaces. If it is nonempty and bounded, it is a *polytope*.

Property 3:

$$P = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$$

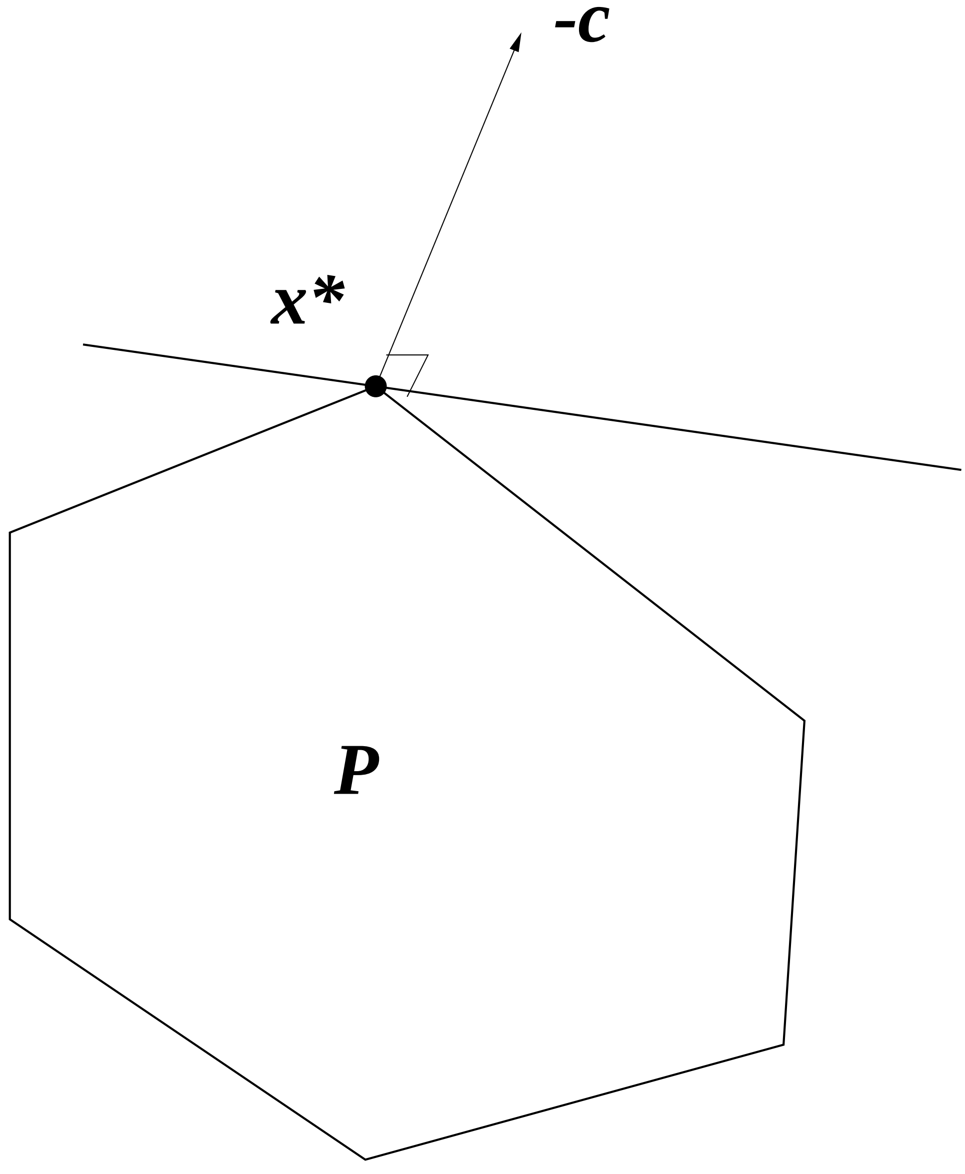
is a polyhedron.

Property 4:

If $P \neq \emptyset$ and $\exists \beta \in \mathbf{R}$ such that

$$P \subset H_L := \{\mathbf{x} \in \mathbf{R}^n | -\mathbf{c}^T\mathbf{x} \leq \beta\},$$

then $\underset{x \in P}{\text{Min}}\, \mathbf{c}^T\mathbf{x} \geq -\beta$



Moreover, if $\mathbf{x}^* \in P \bigcap H$ then $\mathbf{x}^* \in P^*$.

Example:

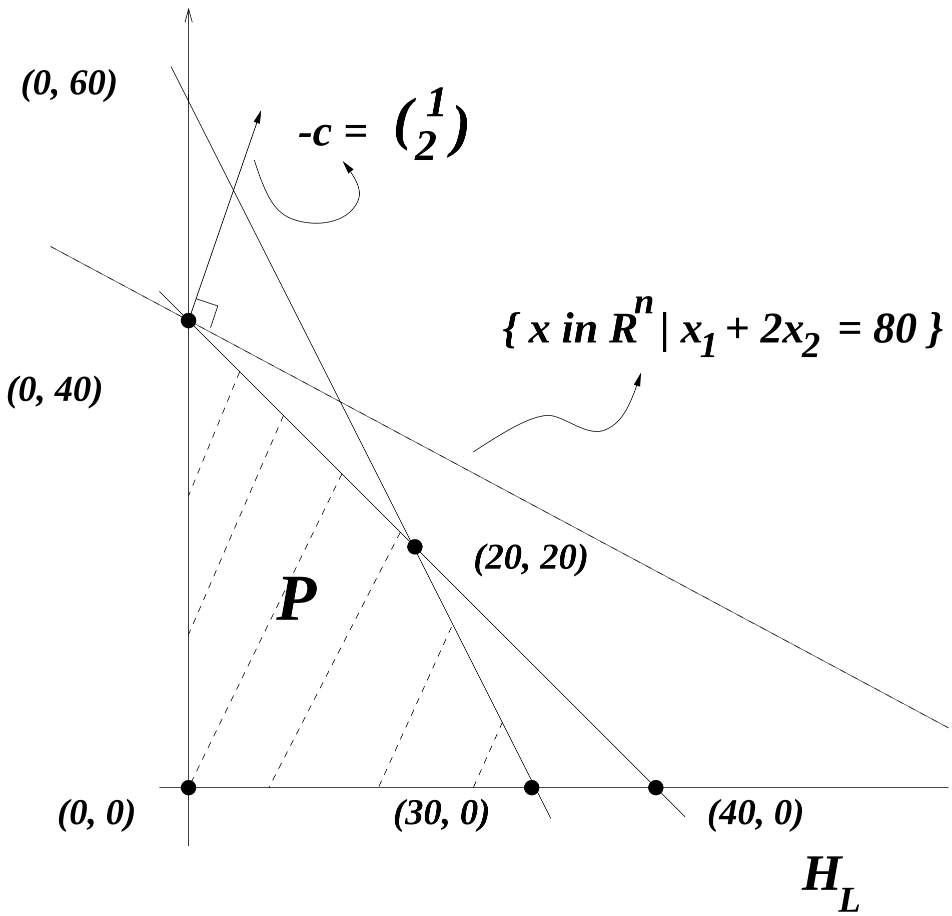
$$
\begin{array}{lrcrl}
\text{Minimize} & -x_1 & - & 2x_2 & \\
\text{s. t.} & x_1 & + & x_2 & \leq 40 \\
 & 2x_1 & + & x_2 & \leq 60 \\
 & x_1, & & x_2, & \geq 0
\end{array}
$$

Standard Form:

$$
\begin{array}{lrcrcrcrl}
\text{Minimize} & -x_1 & - & 2x_2 & & & & & \\
\text{s. t.} & x_1 & + & x_2 & + & x_3 & & & = 40 \\
 & 2x_1 & + & x_2 & & & + & x_4 & = 60 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq 0
\end{array}
$$

$$
\mathbf{c} = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 40 \\ 60 \end{pmatrix}
$$

Solution:



Since $\underset{x \in P}{\mathrm{Min}}\, \mathbf{c}^T \mathbf{x} \geq -80$

also $-x_1 - 2x_2 = -80$ at $\binom{0}{40}$

Hence $\begin{cases} x_1 = 0 \\ x_2 = 40 \end{cases}$ is an optimal solution.

□ Graphic Method:

Step 1: Draw the feasible domain P.
(If $P = \emptyset$, STOP! No solution.)

Step 2: Use $-\mathbf{c}$ as normal vector at each vertex to see if $P \in H_L := \{\mathbf{x} \in \mathbf{R}^n | -\mathbf{c}^T\mathbf{x} \leq \beta\}$ for some $\beta \in \mathbf{R}$.

1. If the answer is "YES", we find an optimal solution.
2. If all answers are "NO", the problem is unbounded below.

□ Affine, Convex Sets , and Cones

- Definition:

$$\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^p \quad \in \mathbf{R}^n,$$
$$\lambda_1, \lambda_2, \ldots, \lambda_p \qquad \in \mathbf{R},$$

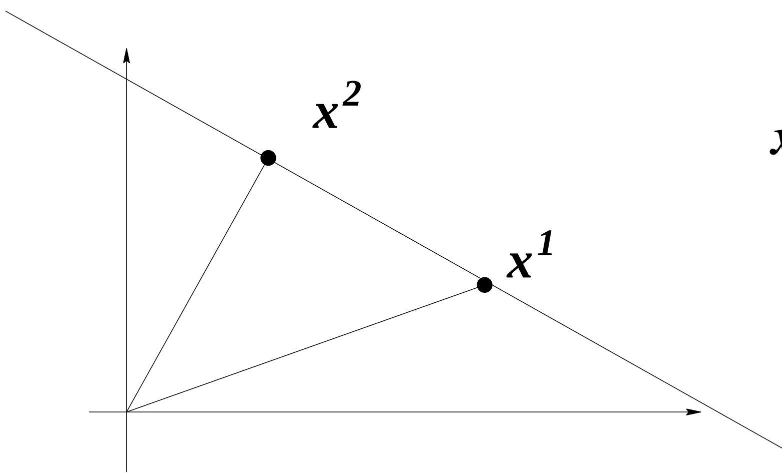
$\mathbf{x} = \sum_{i=1}^{p} \lambda_i \mathbf{x}^i = \lambda_1 \mathbf{x}^1 + \lambda_2 \mathbf{x}^2 + \cdots + \lambda_p \mathbf{x}^p$

is a *linear combination* of $\{\mathbf{x}^1, \ldots, \mathbf{x}^p\}$;

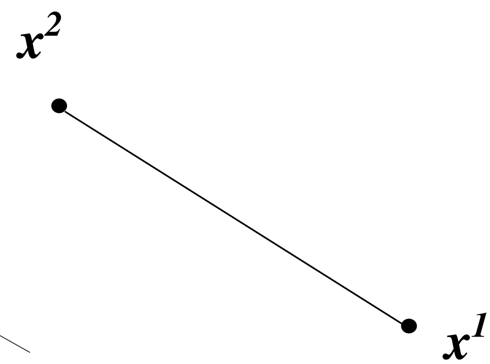
an *affine combination* if $\sum_{i=1}^{p} \lambda_i = 1$;

a *conical combination* if $\lambda_i \geq 0$;

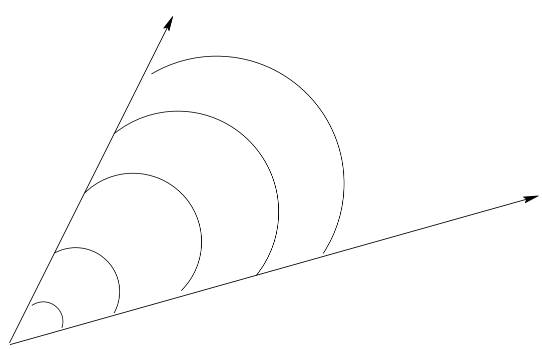
a *convex combination* if $\sum_{i=1}^{p} \lambda_i = 1, \lambda_i \geq 0$.



Affine combination



Convex combination



Conical combination

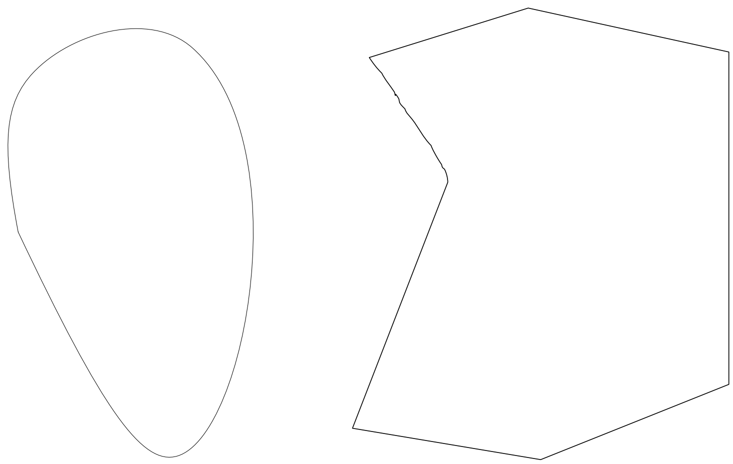
- <u>Definition:</u>

$S \subset \mathbf{R}^n$
is <u>*affine*</u> if
$\forall\ \mathbf{x}^1, \mathbf{x}^2 \in S$, any affine combination of
$\{\mathbf{x}^1, \mathbf{x}^2\} \in S$;

is <u>*convex*</u> if
$\forall\ \mathbf{x}^1, \mathbf{x}^2 \in S$, any convex combination of
$\{\mathbf{x}^1, \mathbf{x}^2\} \in S$;

is a <u>*cone*</u> if
$\forall\ \mathbf{x} \in S,\ \lambda \mathbf{x} \in S$ for $\lambda \geq 0$.

Example:

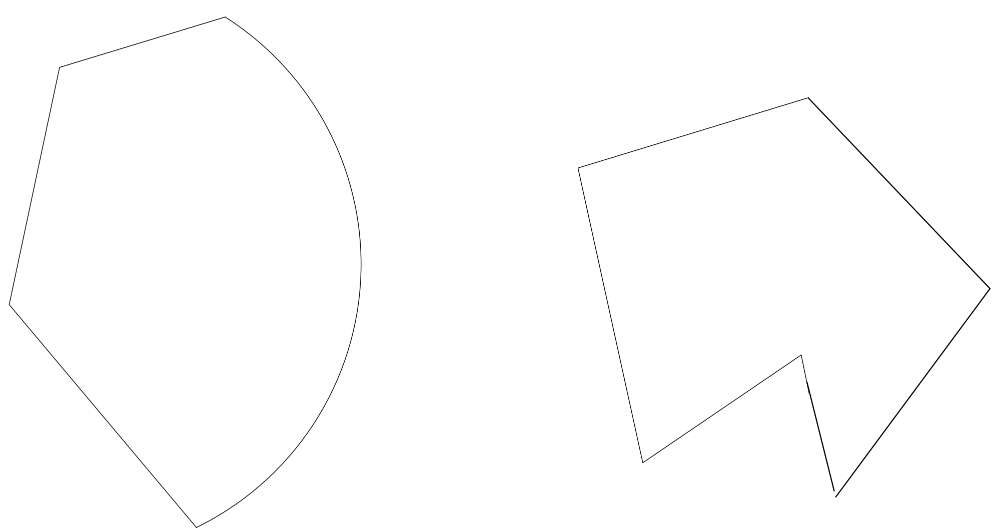


$$
\begin{aligned}
H = &\quad \{\mathbf{x} \in \mathbf{R}^n | \mathbf{a}^T \mathbf{x} = \beta\} \\
H_L = &\quad \{\mathbf{x} \in \mathbf{R}^n | \mathbf{a}^T \mathbf{x} \leq \beta\} \\
&\quad \{\mathbf{x} \in \mathbf{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b}\} \\
P = &\quad \{\mathbf{x} \in \mathbf{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}
\end{aligned}
$$

$$
\begin{array}{cccccc}
\mathrm{C} & \cancel{\mathrm{A}} & \quad & \quad & \cancel{\mathrm{C}} & \cancel{\mathrm{A}} \\
\mathrm{C} & \mathrm{A} & & & \mathrm{C} & \cancel{\mathrm{A}} \\
\mathrm{C} & \mathrm{A} & & & \mathrm{C} & \cancel{\mathrm{A}}
\end{array}
$$

- <u>Definition:</u>

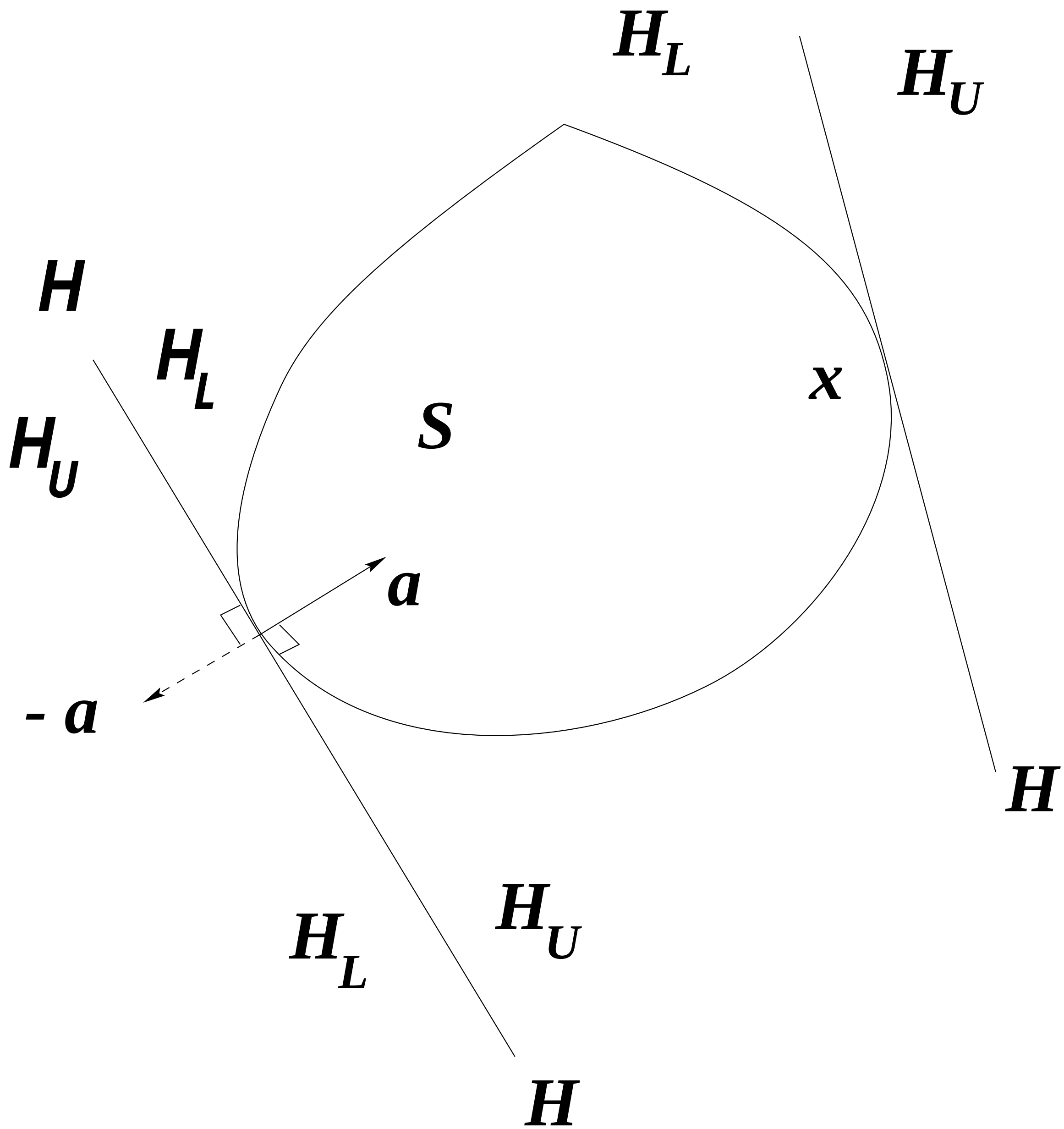
For $S \subset \mathbf{R}^n, \mathbf{x} \in S$ is an <u>*interior point*</u> of S if $\exists\ \epsilon > 0$ such that
$B = \{\mathbf{y} \in \mathbf{R}^n |\ \|\mathbf{x} - \mathbf{y}\| \leq \epsilon\} \subset S$.
Otherwise, $\mathbf{x} \in S$ is a <u>*boundary point*</u> of S.



$$
\begin{aligned}
int(S) = &\quad \{\ \mathbf{x} \text{ is an interior point of } S\ \} \\
bdry(S) = &\quad \{\ \mathbf{x} \text{ is an boundary point of } S\ \}
\end{aligned}
$$

Separation Theorem:

$S \subset \mathbf{R}^n$ is convex, then $\forall \mathbf{x} \in bdry(S), \exists$ a hyperplane H such that $\mathbf{x} \in H$ and either $S \subseteq H_L$ or $S \subseteq H_U$.



Question: Can you now see that if a LP (in two or three dimensions) has a finite optimal solution, then one vertex of $\mathbf{P}$ is optimal ?

[Hint:] Consider the supporting hyperplane

$$H = \{\mathbf{x} \in \mathbf{R}^n | -\mathbf{c}^T\mathbf{x} = \beta\}$$

Question: How about higher dimension?

Answer: Fundamental Theorem of LP!

Special Cone

Given an $m \times n$ matrix $\mathbf{A} = \big(\mathbf{A}_1 | \mathbf{A}_2 | \cdots | \mathbf{A}_n\big)$,

$$A_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \leftarrow j\text{th column of } A.$$

for $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbf{R}^n$ and $\mathbf{x} \geq 0$.

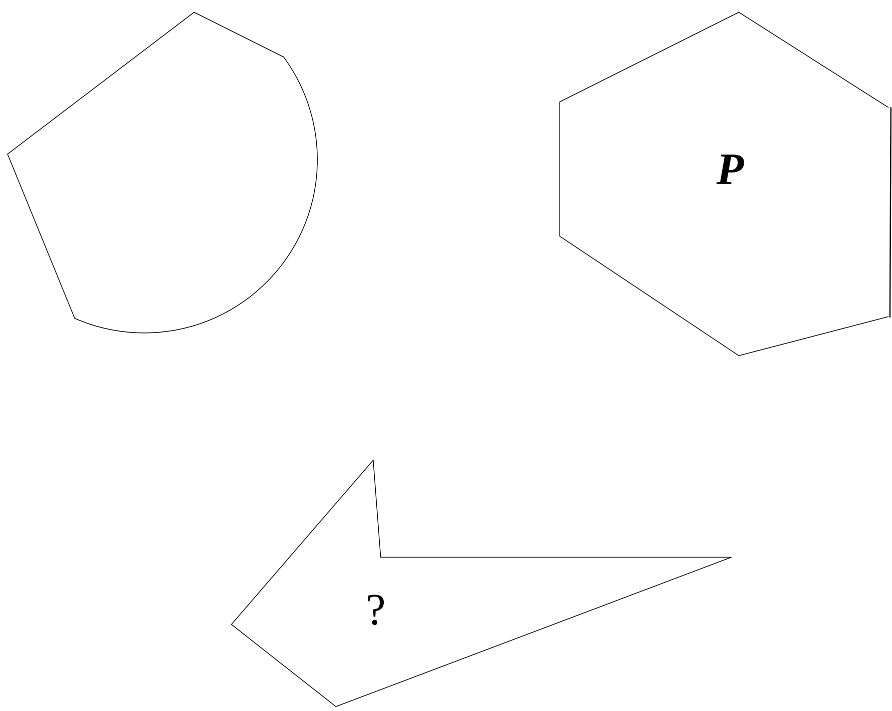
$$\mathbf{y} = \mathbf{A}\mathbf{x} = \big(\mathbf{A}_1 | \mathbf{A}_2 | \cdots | \mathbf{A}_n\big) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ w_n \end{pmatrix} =$$

$\sum_{j=1}^{n} x_j \mathbf{A}_j \in \mathbf{R}^m$.

$\mathbf{A}_c = \{\mathbf{y} \in \mathbf{R}^m | \mathbf{y} = \mathbf{A}\mathbf{x}, \mathbf{x} \in \mathbf{R}^n, \mathbf{x} \geq 0\}$ is a convex cone generated by the columns of $\mathbf{A}$.

- <u>Definition:</u>

$\mathbf{x}$ is an <u>*extreme point*</u> of a convex set $S \subseteq \mathbf{R}^n$, if $\mathbf{x} \in S$ can not be expressed as a convex combination of other points in S.



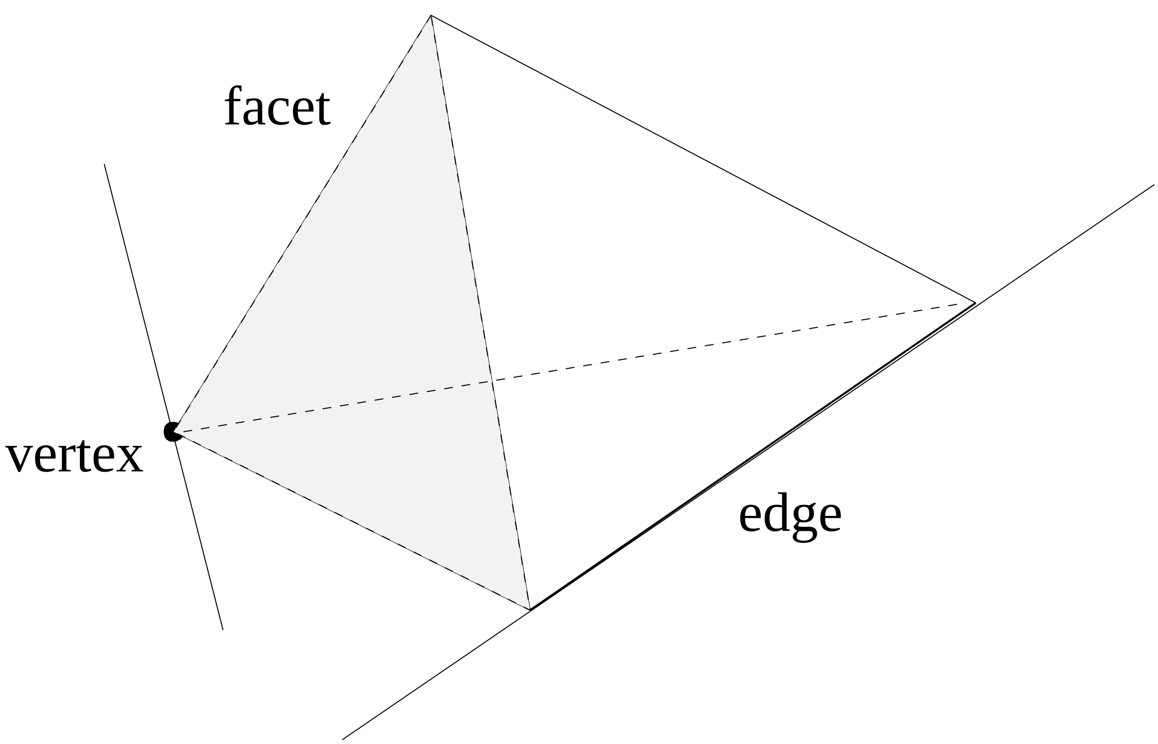
- <u>Definition:</u>

Let P be a convex polyhedron and H be a supporting hyperplane of P, then $F = P \bigcap H$ defines a <u>*face*</u> of P.

When $\dim(F) = 0$, it is a <u>*vertex*</u>

$\dim(F) = 1$, it is an <u>*edge*</u>

$\dim(F) = \dim(P) - 1$, a <u>*facet*</u>



Property:

Let P be a convex polyhedron, $\mathbf{x} \in P$ is a vertex if and only if $\mathbf{x}$ is an extreme point of P.

Representing Extreme Points:

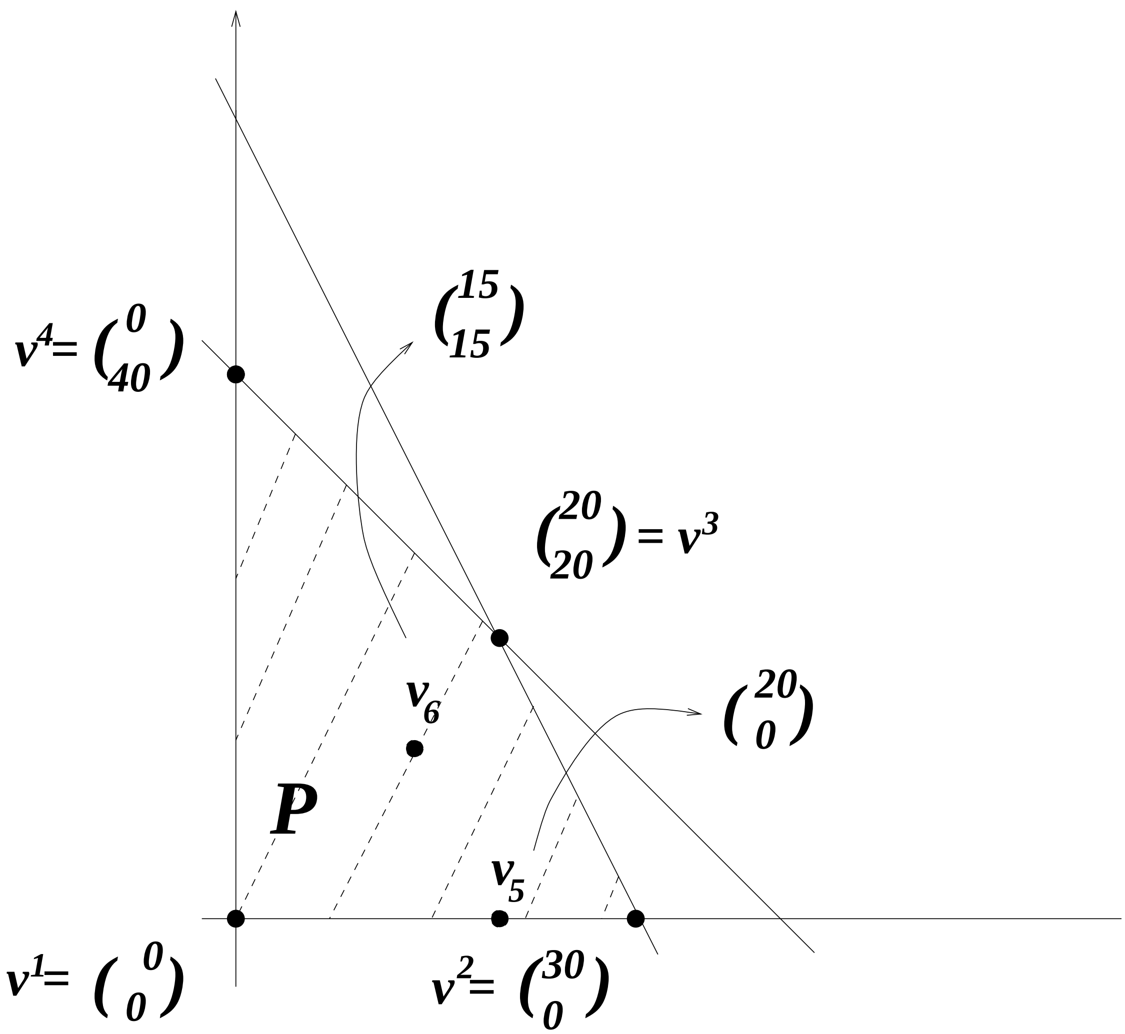
$\mathbf{x}$ is an extreme point of P, then $\mathbf{x}$ is of course a feasible solution of

$$\begin{cases} \mathbf{A}\mathbf{x} = \mathbf{b} \\ \quad \mathbf{x} \geq 0 \end{cases}$$

But <u>what's special of being an extreme point?</u> (in terms of feasible solution).

Example:

$$
\begin{array}{rlrr}
\text{Minimize} & x_1 - 2x_2 & & \\
\text{subject to} & x_1 + x_2 + x_3 & & = 40 \\
& 2x_1 + x_2 & +x_4 & = 60 \\
& x_1, x_2, x_3, x_4 \geq 0. & &
\end{array}
$$



$$v^1 = \begin{pmatrix} 0 \\ 0 \\ 40 \\ 60 \end{pmatrix}, v^2 = \begin{pmatrix} 30 \\ 0 \\ 10 \\ 0 \end{pmatrix}, v^3 = \begin{pmatrix} 20 \\ 20 \\ 0 \\ 0 \end{pmatrix}, v^4 = \begin{pmatrix} 0 \\ 40 \\ 0 \\ 20 \end{pmatrix}.$$

$$v^5 = \begin{pmatrix} 20 \\ 0 \\ 20 \\ 20 \end{pmatrix} \leftarrow \text{one zero } x_i.$$

$$v^6 = \begin{pmatrix} 15 \\ 15 \\ 10 \\ 15 \end{pmatrix} \leftarrow \text{no zero } x_i.$$

$n = 4,\ m = 2,\ n - m = 2$

Observation:

An extreme point of P is obtained by setting $n - m$ variables to be zero and solving the remaining m variables in m equations. Therefore, the columns of $\mathbf{A}$ corresponding to the non-zero (positive) variables better be linear independent!

$$\begin{cases} x_1 + x_2 + x_3 & & = 40 \\ 2x_1 + x_2 & +x_4 & = 60 \\ x_1, x_2, x_3, x_4 \geq 0. & & \end{cases}$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} x_1 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_4 = \begin{pmatrix} 40 \\ 60 \end{pmatrix}.$$

Exterme Points & Basic Feasible Solutions

Theorem 2.1:

A point $\mathbf{x} \in P = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{A}\mathbf{x} = b, \mathbf{x} \geq 0\}$ is an extreme point of P iff the columns of $\mathbf{A}$ corresponding to positive components of $\mathbf{x}$ are linearly independent.

Proof: WLOG, assume that the first p components of $\mathbf{x}$ are positive and the rest are zeroes, *i.e.*

$$\mathbf{x} = \begin{pmatrix} \bar{\mathbf{x}} \\ \mathbf{0} \end{pmatrix} \text{ where } \bar{\mathbf{x}} = \begin{pmatrix} x_1 \\ \vdots \\ x_p \end{pmatrix} > \mathbf{0}.$$

also denote the first p columns of $\mathbf{A}$ by $\bar{\mathbf{A}}$, then $\mathbf{A}\mathbf{x} = \bar{\mathbf{A}}\bar{\mathbf{x}} = \mathbf{b}$.

Suppose that the columns of $\bar{\mathbf{A}}$ are not linearly independent, then $\exists\ \bar{\mathbf{w}} \neq \mathbf{0}$ such htat $\bar{\mathbf{A}}\bar{\mathbf{w}} = 0$.

Notice that for ϵ is small enough

$\bar{\mathbf{x}} \pm \epsilon\bar{\mathbf{w}} \geq \mathbf{0}$ and $\bar{\mathbf{A}}(\bar{\mathbf{x}} \pm \epsilon\bar{\mathbf{w}}) = \bar{\mathbf{A}}\bar{\mathbf{x}} = \mathbf{b}$

Hence

$$\mathbf{y}^1 = \begin{pmatrix} \bar{\mathbf{x}} + \epsilon \bar{\mathbf{w}} \\ \mathbf{0} \end{pmatrix} \in P$$

$$\mathbf{y}^2 = \begin{pmatrix} \bar{\mathbf{x}} - \epsilon \bar{\mathbf{w}} \\ \mathbf{0} \end{pmatrix} \in P$$

and $\mathbf{x} = \frac{1}{2}\mathbf{y}^1 + \frac{1}{2}\mathbf{y}^2$, *i.e.* $\mathbf{x}$ can not be a vertex (extreme point) of P.
Thus, $\mathbf{x}$ is an exterme point $\Rightarrow$ columns of $\bar{\mathbf{A}}$ are linearly independent.

Suppose that $\mathbf{x}$ is not an extreme point, then
$\mathbf{x} = \lambda \mathbf{y}^1 + (1-\lambda)\mathbf{y}^2$ for some
$\mathbf{y}^1, \mathbf{y}^2 \in P,\ \mathbf{y}^1 \neq \mathbf{y}^2$ and $0 < \lambda < 1$,
Since $\mathbf{y}^1 \geq 0, \mathbf{y}^2 \geq 0$ and $0 < \lambda < 1$.
the last $n-p$ components of $\mathbf{y}^1$ must be zero, *i.e.*

$$\mathbf{y}^1 = \begin{pmatrix} \bar{\mathbf{y}}^1 \\ \mathbf{0} \end{pmatrix}$$

Now

$$\mathbf{x} - \mathbf{y}^1 = \begin{pmatrix} \bar{\mathbf{x}} - \bar{\mathbf{y}}^1 \\ \mathbf{0} \end{pmatrix} \neq \mathbf{0}$$

and $\mathbf{A}(\mathbf{x} - \mathbf{y}^1) = \mathbf{A}\mathbf{x} - \mathbf{A}\mathbf{y}^1 = \mathbf{b} - \mathbf{b} = \mathbf{0}$

$\Rightarrow$ columns of $\mathbf{A}$ are linearly dependent.

Thus , columns of $\bar{\mathbf{A}}$ are linearly independent

$\Rightarrow$ $\mathbf{x}$ is an extreme point.

Definition: $\mathbf{A}$ $m \times n$ matrix ($m \leq n$).

$\mathbf{A}$ has *full (row) rank* if $\exists$ m linearly independent columns of $\mathbf{A}$.
In this case, rearrange

$$\begin{array}{ccccccc} \mathbf{A} & = & (& \mathbf{B} & | & \mathbf{N} &) \\ & & & \uparrow & & \uparrow & \\ & & & \text{basis} & & \text{non} - \text{basis} & \end{array}$$

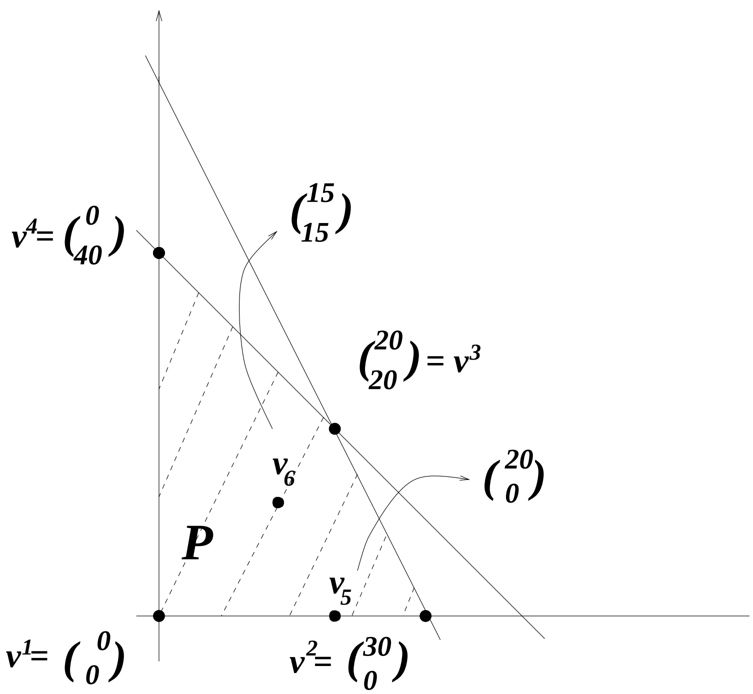
$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_N \end{pmatrix} \begin{array}{l} \leftarrow \text{ basic variables} \\ \leftarrow \text{ non-basic variables} \end{array}$$

If we set $\mathbf{x}_N = 0$ and solve $\mathbf{x}_B$ for $\mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{x}_B = \mathbf{b}$ then $\mathbf{x}$ is a *basic solution* (bs).

Furthermore, if $\mathbf{x}_B \geq 0$, then $\mathbf{x}$ is a *basic feasible solution* (bfs).

Example:

$$\begin{array}{rlr} \text{Minimize} & x_1 - 2x_2 & \\ \text{subject to} & x_1 + x_2 + x_3 & = 40 \\ & 2x_1 + x_2 \qquad\qquad + x_4 & = 60 \\ & x_1, x_2, x_3, x_4 \geq 0. & \end{array}$$



$$v^1 = \begin{pmatrix} 0 \\ 0 \\ 40 \\ 60 \end{pmatrix}, v^2 = \begin{pmatrix} 30 \\ 0 \\ 10 \\ 0 \end{pmatrix}, v^3 = \begin{pmatrix} 20 \\ 20 \\ 0 \\ 0 \end{pmatrix}, v^4 = \begin{pmatrix} 0 \\ 40 \\ 0 \\ 20 \end{pmatrix} \text{bfs}$$

$$v^5 = \begin{pmatrix} 20 \\ 0 \\ 20 \\ 20 \end{pmatrix}, v^6 = \begin{pmatrix} 15 \\ 15 \\ 10 \\ 15 \end{pmatrix}, v^7 = \begin{pmatrix} 40 \\ 0 \\ 0 \\ -20 \end{pmatrix}, v^8 = \begin{pmatrix} 0 \\ 60 \\ -20 \\ 0 \end{pmatrix} \text{bs}$$

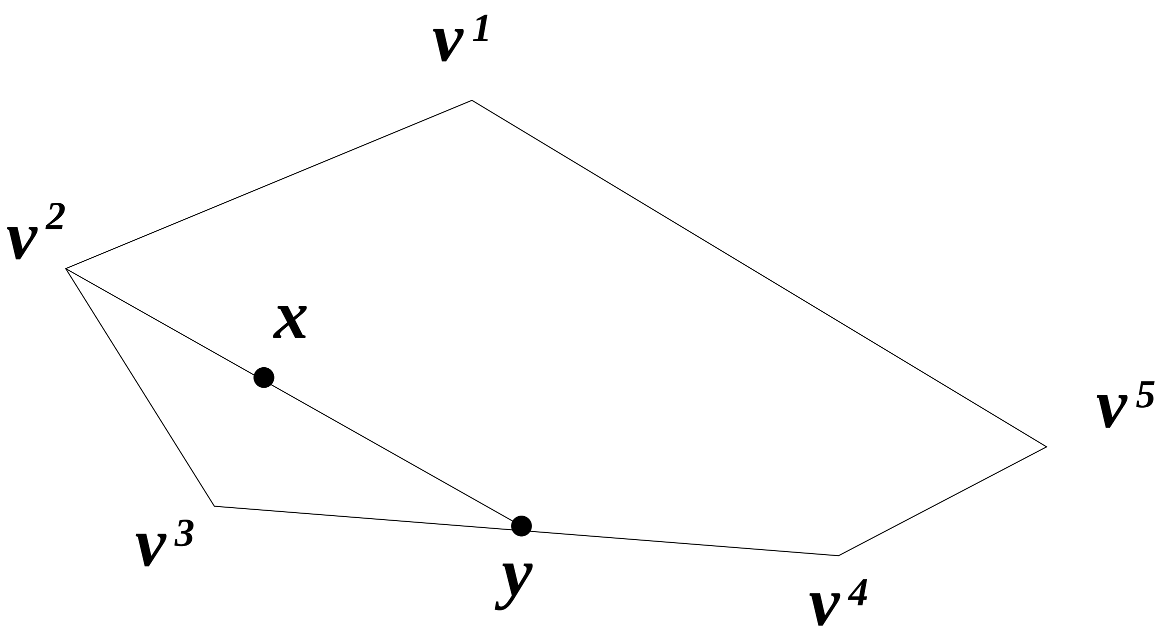
When $\mathbf{A}$ does not have full rank, then either
(1) $\mathbf{Ax} = \mathbf{b}$ has no solution and $P = \emptyset$, or
(2) some constraints are *redundant.*
For (2), after removing the redundant constraints, new $\mathbf{A}$ has full rank.

Corollary 2.1.1: A point $\mathbf{x} \in P$ is an extreme point of P *iff* $\mathbf{x}$ is a bfs corresponding to some basis $\mathbf{B}$.

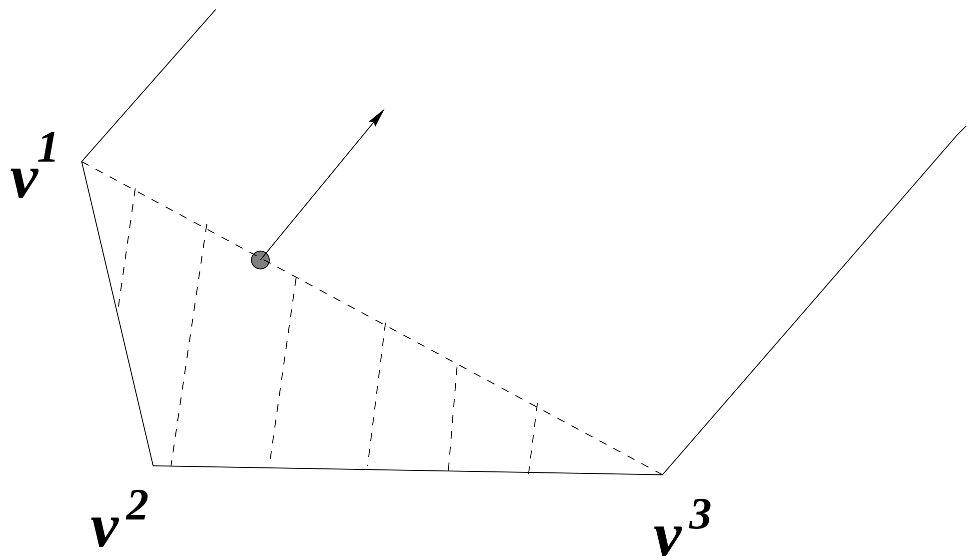
Corollary 2.1.2: The polyhedron P has only a finite number of extreme points.
Proof: # of ways to choose m linearly independent columns from n columns
$\leq \; C(n, m) = \frac{n!}{m!(n-m)!}$.

Resolution Theorem

Observation: When $P = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$ is a polytope, then any point in P can be represented as a convex combination of the extreme points of P.



Question: How is the general case?



<u>Definition:</u> $\mathbf{d}\ (\neq \mathbf{0}) \in \mathbf{R}^n$ is an *<u>extremal direction</u>* of P if $\forall\ \mathbf{x}^0 \in P$, $\{\mathbf{x} \in \mathbf{R}^n | \mathbf{x} = \mathbf{x}^0 + \lambda\mathbf{d},\ \lambda \geq 0\} \subset P$.

<u>Observation:</u>

(1) P is unbounded $\Leftrightarrow$ P has an extremal direction.

(2) $\mathbf{d}\ (\neq \mathbf{0})$ is an extremal direction of P $\Leftrightarrow$ $\mathbf{Ad} = 0$ and $\mathbf{d} \geq \mathbf{0}$

Theorem 2.2 Resolution Theorem:

Let $V = \{v^i \in \mathbf{R}^n | i \in I\}$ be a set of all exterme points of P, I is a finite index set, then $\forall\ \mathbf{x} \in P$, we have

$$\mathbf{x} = \sum_{i \in I} \lambda_i v^i + \mathbf{d}$$

where

$$\sum_{i \in I} \lambda_i = 1,\ \ \lambda_i \geq 0\ \forall i \in I.$$

and either $\mathbf{d} = \mathbf{0}$ or

$\mathbf{d}$ is an extermal direction of P.

Corollary 2.2.1: If P is bounded, then any $\mathbf{x} \in P$ can be expressed as a convex combination of its extreme points.

Corollary 2.2.2: If $P \neq \emptyset$, then P has at least one extreme point.

Fundamental Theorem of LP

Theorem 2.3

If $P \neq \emptyset$, then the minimun objective value of $\mathbf{z} = \mathbf{c}^T\mathbf{x}$ over P is either unbounded below or attained at an extreme point of P.

Proof: By the resolution theorem, there are two cases:

Case 1: P has an extremal direction $\mathbf{d}$ such that $\mathbf{c}^T\mathbf{d} < \mathbf{0}$. Hence P is unbounded and $\mathbf{z} \to -\infty$ along $\mathbf{d}$.

Case 2: P does not have any extremal direction $\mathbf{d}$ such that $\mathbf{c}^T\mathbf{d} < 0$, then $\forall\ \mathbf{x} \in P$, either $\mathbf{x} = \sum_{i\in I} \lambda_i v^i$ with $\sum_{i\in I} \lambda_i = 1,\ \lambda_i \geq 0$, or $\mathbf{x} = \sum_{i\in I} \lambda_i v^i + \bar{\mathbf{d}}$ with $\mathbf{c}^T\bar{\mathbf{d}} \geq \mathbf{0}$.
In both cases,

$$
\begin{aligned}
\mathbf{c}^T\mathbf{x} &= \mathbf{c}^T[\textstyle\sum_{i\in I}\lambda_i v^i](+\mathbf{c}^T\bar{\mathbf{d}}) \\
&\geq \textstyle\sum_{i\in I}\lambda_i(\mathbf{c}^T v^i) \\
&\geq \min_{i\in I}\{\mathbf{c}^T v^i\}(\textstyle\sum_{i\in I}\lambda_i) \\
&= \min_{i\in I}\{\mathbf{c}^T v^i\} \\
&= \mathbf{c}^T v^{\min}.
\end{aligned}
$$

Hence the minimum of $\mathbf{z}$ is attained at one extreme point!