OR/IE/MA 505

Standard Form Linear Program (LP)

- n variables:
  $x_1, x_2, \ldots, x_n$
- 1 objective function to be minimized:
  $\mathbf{z} = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$
- m constraints:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

- non-negativity requirements:
  $x_1 \geq 0,\ x_2 \geq 0, \cdots,\ x_n \geq 0$

Matrix Form

$$\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

cost vector, right-hand-side vector

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

constraint matrix

$$\begin{aligned} \text{Min } & \mathbf{c}^T\mathbf{x} \\ \text{s. t. } & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

Minimize $\mathbf{z} = c_1x_1 + c_2x_2 + \cdots + c_nx_n$
subject to

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\qquad\qquad\vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\
&x_1 \geq 0,\ x_2 \geq 0, \cdots,\ x_n \geq 0
\end{aligned}
$$

1. Minimize objective function
2. Equality constraints
3. Non-negative variables

Example: Meat Transportation

Supply	tons
NC	4,000
OK	3,000
IO	2,500
VA	1,500

Demand	tons
NY	5,000
LA	2,000
DC	4,000

Cost($)	NY	LA	DC
NC	30	10	40
OK	15	20	60
IO	60	35	25
VA	5	45	75

Question:

How to satisfy customers in a cost effective manner?

1. What are variables to be determined?
2. What is the objective function?
3. What are constraints?

(1) Variables

	NY	LA	DC
NC	x_{11}	x_{12}	x_{13}
OK	x_{21}	x_{22}	x_{23}
IO	x_{31}	x_{32}	x_{33}
VA	x_{41}	x_{42}	x_{43}

(2) Objective Function

$$\begin{aligned}\mathbf{z} = \quad & 30x_{11} + 10x_{12} + 40x_{13} \\ + \quad & 15x_{21} + 20x_{22} + 60x_{23} \\ + \quad & 60x_{31} + 35x_{32} + 25x_{33} \\ + \quad & 5x_{41} + 45x_{42} + 75x_{43}\end{aligned}$$

(3) Constraints (one redundant)

$$\begin{aligned}\text{(supply)} \quad & x_{11} + x_{12} + x_{13} && = 4,000 \\ & x_{21} + x_{22} + x_{23} && = 3,000 \\ & x_{31} + x_{32} + x_{33} && = 2,500 \\ & x_{41} + x_{42} + x_{43} && = 1,500\end{aligned}$$

$$
\begin{aligned}
\text{(demand)} \quad & x_{11} + x_{21} + x_{31} + x_{41} = 5,000 \\
& x_{12} + x_{22} + x_{32} + x_{42} = 2,000 \\
& x_{13} + x_{23} + x_{33} + x_{43} = 4,000
\end{aligned}
$$

LP Model

$$
\begin{aligned}
\text{Minimize } \mathbf{z} = \quad & 30x_{11} + 10x_{12} + 40x_{13} \\
+ \quad & 15x_{21} + 20x_{22} + 60x_{23} \\
+ \quad & 60x_{31} + 35x_{32} + 25x_{33} \\
+ \quad & 5x_{41} + 45x_{42} + 75x_{43}
\end{aligned}
$$

$$
\begin{aligned}
\text{subject to} \quad & x_{11} + x_{12} + x_{13} && = && 4,000 \\
& x_{21} + x_{22} + x_{23} && = && 3,000 \\
& x_{31} + x_{32} + x_{33} && = && 2,500 \\
& x_{41} + x_{42} + x_{43} && = && 1,500 \\
& x_{11} + x_{21} + x_{31} + x_{41} && = && 5,000 \\
& x_{12} + x_{22} + x_{32} + x_{42} && = && 2,000 \\
& x_{13} + x_{23} + x_{33} + x_{43} && = && 4,000 \\
& x_{11}, x_{12}, \ldots, x_{33}, x_{43} \geq 0
\end{aligned}
$$

Embedded Assumptions

1. Proportionality Assumption

   No discount.

   No economy of return to scale.

2. Additivity Assumption

   Total contribution = $\sum_{\text{variables}}$ individual contribution.

3. Divisibility Assumption

   Any fractional value is allowed.

4. Certainty Assumption

   Each parameter is known for sure.

Converting to the "standard form"

Example:

$$
\begin{array}{lrcrcrl}
\text{Maximize} & 3x_1 & - & 2x_2 & - & 4|x_3| & \\
\text{s. t.} & -x_1 & + & 2x_2 & & & \leq -5 \\
 & & & 3x_2 & - & x_3 & \geq 6 \\
 & 2x_1 & & & + & x_3 & = 12 \\
 & x_1, & & x_2 & & \geq 0 &
\end{array}
$$

Rule 1: Unrestricted (free) variables

$$x_i \in R \qquad\qquad x_i^+ = \begin{cases} x_i, & \text{if } x_i \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\Updownarrow$$

$$x_i^- = \begin{cases} 0, & \text{if } x_i \geq 0 \\ -x_i, & \text{otherwise} \end{cases}$$

$$x_i = x_i^+ - x_i^-$$

$$x_i^+,\ x_i^- \geq 0$$

also $|x_i| = x_i^+ + x_i^-$

(Note that the condition of $x_i^+ \times x_i^- = 0$ may be relaxed.)

$$
\begin{array}{lrcrcrl}
\text{Maximize} & 3x_1 & - & 2x_2 & - & 4(x_3^+ + x_3^-) & \\
\text{s. t.} & -x_1 & + & 2x_2 & & & \leq -5 \\
 & & & 3x_2 & - & (x_3^+ - x_3^-) & \geq 6 \\
 & 2x_1 & & & + & (x_3^+ - x_3^-) & = 12 \\
 & x_1, & & x_2, & & x_3^+,\ x_3^-, & \geq 0
\end{array}
$$

Rule 2: Inequality constraint

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \qquad \le b_i$$

$\Updownarrow$ add a *slack variable* $s_i \ge 0$

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n + s_i \qquad = b_i$$

$$s_i \ge 0$$

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \qquad \ge b_i$$

$\Updownarrow$ subtract an *excess variable* $e_i \ge 0$

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - e_i \qquad = b_i$$

$$e_i \ge 0$$

Maximize

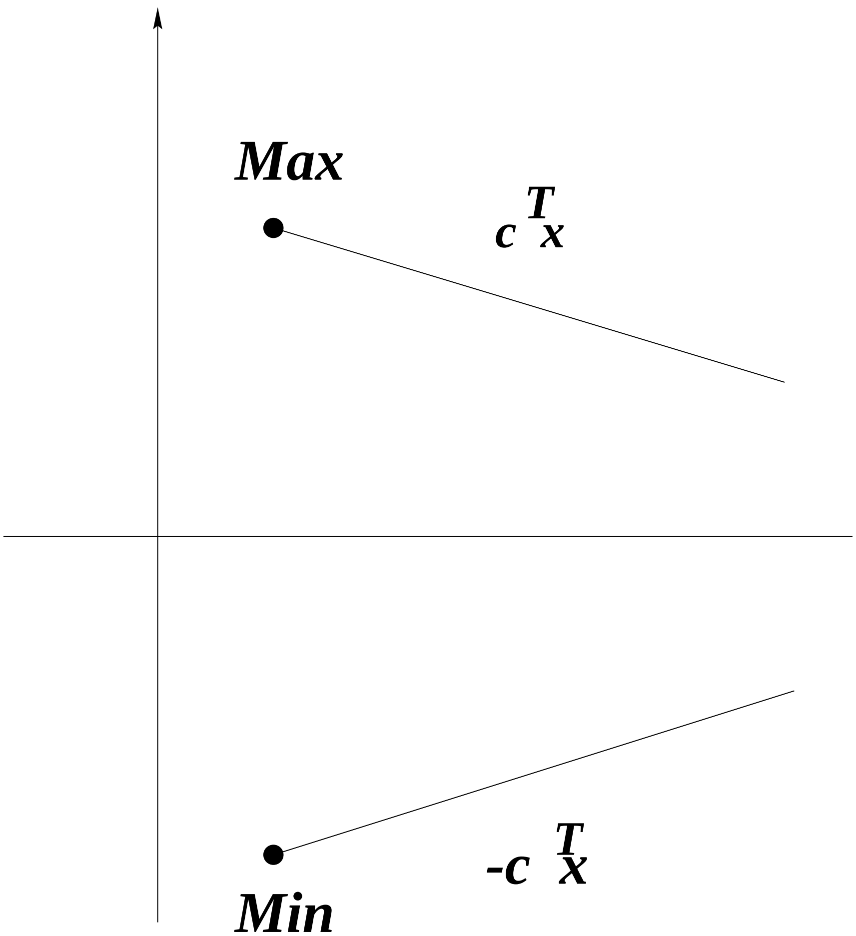
$$3x_1 - 2x_2 - 4x_3^+ - 4x_3^-$$

subject to

$$\begin{array}{rcccccl} -x_1 & + & 2x_2 & & +x_4 & & = -5 \\ & & 3x_2 & -x_3^+ + x_3^- & & -x_5 & = 6 \\ 2x_1 & & & +x_3^+ - x_3^- & & & = 12 \\ x_1, & & x_2, & x_3^+,\ x_3^- & x_4, & x_5 & \geq 0 \end{array}$$

Rule 3: Minimization of the objective function

$$\text{Max } c_1x_1 + c_2x_2 + \cdots + c_nx_n$$



Max $\boldsymbol{c^T x = -\text{Min}(-c^T x)}$

(−)Minimize

$$
\begin{array}{rrrrrr}
-3x_1 & +2x_2 & +4x_3^+ + 4x_3^- & & & \\
\text{subject to} & & & & & \\
-x_1 & +2x_2 & & +x_4 & & = -5 \\
 & 3x_2 & -x_3^+ + x_3^- & & -x_5 & = 6 \\
2x_1 & & +x_3^+ - x_3^- & & & = 12 \\
x_1, & x_2, & x_3^+,\ x_3^- & x_4, & x_5 & \geq 0
\end{array}
$$