ACTIVITY FLOAT PERFORMANCE THEORY AND ITS APPLICATION UNDER GPRs (Brief introduction)

1. New graphical representation

The new representation is in consistence with the CPM on (AOA) net work, which facilitates the research of the network characteristic.

1.1. “GPRs with lower bounds”

1.1.1. FS relation

The FS relation is “the start time of activity $B$ is no earlier than $k$ days after the finish time of $A$”. The graphical representation is shown as Fig. 1 by adding a corresponding relation activity “$i \xrightarrow{k} j$”.

![Fig.1](image1)

1.1.2. SS relation

The SS relation is “the start time of activity $B$ is no earlier than $k$ days after that of $A$”. The graphical representation is shown as Fig. 2 by adding a corresponding relation activity “$i \xrightarrow{H \cap k} j$”.

![Fig.2](image2)

No doubt, the above representation is wrong due to the following reason.

As the latest start (LS) time of activity $B$ is $(35 - 5) = 30$ days as shown in the original CPM network, and “the start time of activity $B$ is no earlier than $k$ days after that of $A$”, the latest start time of $A$ should be $30 - 3 = 27$ days. However, the $LS_A = 39 - 8 = 31 \neq 27$.

Amendment: covert the SS relation into FS relation as Fig.3.
1.1.3. SF relation

The SF relation is “the finish time of activity B is no earlier than $k$ days after the start time of $A$”. The graphical representation is shown as Fig. 4 by adding a corresponding relation activity $i \rightarrow j$.

No doubt, the above representation is wrong due to the following reason. The earliest start time of activity $A$ is 20. As “the finish time of activity $B$ is no earlier than 7 days after the start time of $A$”, the earliest finish time of $B$ should be $20 + 7 = 27$ days. However, the earliest finish time of $B$ is $EF_b = 21 + 4 = 25 \neq 27$ determined by $(b_1, j)$.

Amendment: covert the SF relation into FS relation as Fig. 5.
1.1.4. *FF* relation

The *FF* relation is “the finish time of activity $B$ is no earlier than $k$ days after that of $A$”. The graphical representation is shown as Fig.6 by adding a corresponding relation activity $i \rightarrow j$.

![Graphical representation of FF relation](image)

No doubt, the above representation is wrong due to the following reason.

The earliest finish time of activity $A$ is $EF_A = 20 + 8 = 28$. As “the finish time of activity $B$ is no earlier than 1 days after that of $A$”, the earliest finish time of $B$ should be $28 + 1 = 29$ days. However, the earliest finish time of $B$ is $EF_B = 21 + 4 = 25 \neq 27$ determined by $(b_i, j)$.

Amendment: convert the *FF* relation into *FS* relation as Fig.5.

![Graphical representation of FS relation](image)

1.2. “GPRs with upper bounds”

1.2.1. *SF* relation

The *SF* relation is “The finish time of activity $B$ is no more than $k$ days after the start time of $A$”. The graphical representation is shown as Fig.8 by adding a corresponding relation activity $i \rightarrow j$.
1.2.2. SS relation

The SS relation is “The Start time of activity B is no more than k days after that of A”. The graphical representation is shows as Fig.9 by adding a corresponding relation activity 

No doubt, the above representation is wrong due to the following reason.

The latest start time of activity A is \( LS_a = 39 - 8 = 31 \) determined by \((i, a_2)\). As The Start time of activity B is no more than 1 days after that of A”, the latest start time of B should be \( 31 + 1 = 32 \) days. However, the latest start time of B is \( LS_b = 38 - 4 = 34 \neq 32 \) determined by \((j, b_2)\).

Amendment: covert the SS relation into SF relation as Fig.10.
1.2.3. **FS relation**

The FS relation is “The Start time of activity B is no more than k days after the finish time of A”. The graphical representation is shows as Fig.11 by adding a corresponding relation activity

No doubt, the above representation is wrong due to the following reason.

The latest finish time of activity A is $LF_A = 31$ determined by $(a_i, j)$. As “The Start time of activity B is no more than 1 days after the finish time of A”, the latest start time of B should be $31 + 1 = 32$ days. However, the latest start time of B is $LS_B = 38 - 4 = 34 ≠ 32$ determined by $(j, b_2)$.

Amendment: covert the FS relation into SF relation as Fig.12.
1.2.4. **FF relation**

The **FF** relation is “The finish time of activity \( B \) is no more than \( k \) days after that of \( A \)”. The graphical representation is shows as Fig.13 by adding a corresponding relation activity 

No doubt, the above representation is wrong with the same reason as mentioned before. Amendment: covert the **FF** relation into **SF** relation as Fig.14.
1.3. Generalization

GPRs with lower bounds should be converted into “FS relation” and those with upper bounds should be done into “SF relation”, then they can be represented by CPM methods.

Standard rule: Activity $A$, GPR relation $H$ and activity $B$ should be represented as a chain in order, $A \rightarrow H \rightarrow B$ or $B \rightarrow H \rightarrow A$.

1.4. GPRs with both lower and upper bounds

For example: “the start time of activity $B$ should be no earlier than 3 days, and no more than 6 days after the start time of activity $A$”

The first step: Graphical representation with “$H$ $\rightarrow$” (GPR relation activity with lower bounds) and “$R$ $\leftarrow$” (GPR relation activity with lower bounds) shown as Fig.15.

Fig.15.

The second step: Activity $A$, GPR relation $(ij)$ and activity $B$ should be represented as a chain in order, Fig.16.

Fig.16
2. Time parameters definition and calculation under GPRs

There will be cycles in the network due to the graphical repression of GPRs which is totally different from the CPM network. Therefore, the time parameters in CPM should be defined again under GPRs.

The notations and representations are shown as the follows:

- The start time of node \( (i) \): \( t_i = t_i(E) = S_i = ES_i \)
- The finish time of node \( (i) \): \( F_i = t_i(L) = LF_i \)
- The earliest start time of activity \( (ij) \): \( ES_{ij} = t_y(E) \)
- The earliest finish time of activity \( (ij) \): \( EF_{ij} = ES_{ij} + T_y \)
- The latest finish time of activity \( (ij) \): \( LF_{ij} = t_y(L) \)
- The latest start time of activity \( (ij) \): \( LS_{ij} = LF_{ij} - T_y \)
- The total float: \( S_{ij}^{(1)} = TF_{ij} \)
- The safety float: \( S_{ij}^{(2)} = SF_{ij} \)
- The free float: \( S_{ij}^{(3)} = FF_{ij} \)
- The interference float: \( S_{ij}^{(4)} = IF_{ij} \)

\( \mu \) denote the length of path \( \mu \), \( \mu^\nu \) denote the critical path, \( \mu_{ij}^\nu \) denote the longest path passing activity \( (ij) \), \( \mu_i^\nu \) denote the longest path passing the node \( (i) \), and \( \mu_i^* \) denote the longest path from the node \( (i) \) to the begin node \( (1) \), and \( \mu_w^* \) denote the longest path from the node \( (i) \) to the end node \( (w) \).

2.1. The start time \( (ES_i) \) of node \( (i) \)

2.1.1. Definition

The length \( (\mu_i^*) \) of the longest path \( (\mu_i^*) \) from the node \( (i) \) to the begin node \( (1) \) is defined as the start time \( (ES_i) \) of node \( (i) \), that is
2.1.2. Calculation method

2.1.2.1. Ford method

As there is cycles existed, the Ford method should be used.

2.1.2.2. Multi-step method (new algorithm designed by us)

<1> Cancel the activities with negative length, then the GRPs network turns out to be a CPM network and then the CPM method is used to calculated the $ES_i$.

<2> Add the activities with negative length again, then check (from the end node to the begin node $i$), the earliest start time of all the activities with negative length and backward direction, such as activity $A$ (network $H_{-k}$ network).

<i> If $ES_j + (-k) \leq ES_i$, then $ES'_i = ES_j$;

<i> If $ES_j + (-k) > ES_i$, then $ES'_i = ES_j + (-k)$.

If all the activities match condition <i>, then the algorithm is finished.
If there is at least one activity match condition <ii>, then go to step <3>.

<3> Check (from the begin node $i$ to the end node) all the activities with forward direction, such as activity $A$ (network $A_{-k}$ network) (include lower and upper bounds).

<i> If $ES_j + k \leq ES_i$, then $ES'_i = ES_j$;

<i> If $ES_j + k > ES_i$, then $ES'_i = ES_j + k$.

Then, go back to step <2>.

2.2. The finish time $LF_i$ of node $i$

2.2.1. Definition

The length of the critical path is $\mu = ES_w$, $\mu_i$ denote the length of the longest path from node $i$ to the end node $w$, then the finish time of node $i$, denoted as $LF_i$ can be worked our as the follows:

$$LF_i = \mu - \mu_i$$

2.2.2. Calculation method
2.2.2.1. Ford method

<1> Calculate $ES_w = \mu$ ;

<2> Calculate the $\mu_i^0$ (the length of the longest path from node $(i)$ to the end node $(w)$) using Ford method;

<3> $LF_i = ES_w - \mu_i^0 = \mu - \mu_i^0$.

2.2.2.2. Multi-step method (new algorism designed by us, and there is no positive cycle)

<1> $ES_w = LF_w$.

<2> Calculate the latest time of all the activities with forward direction and positive length (begin from the end node $(w)$), the activities with backward direction and negative length are not considered in this step.

<3> Check the activities with backward direction and negative length, (begin from the begin node $(1)$ to the end node $(w)$), such as activity $(A)$ (network $\xrightarrow{-k}{H}$ network).

<i> If $LF_j - (-k) = LF_j + k \geq LF_j$ , then $LF_j' = LF_j$ ;

<i> If $LF_j - (-k) = LF_j + k < LF_j$ , then $LF_j' = LF_j + k$ .

If all the activities match condition <i>, then the algorism is finished.

If there is at least one activity match condition <ii>, then go to step <4>

<4> Check (from the end node $(w)$ to the begin node $(1)$) all the activities with forward direction and positive length, such as activity $(A)$ (network $\xleftarrow{-k}{H}$ network).

<i> If $LF_j - k = LF_j$ , then $TF_j' = TF_j$ ;

<i> If $LF_j - k < LF_j$ , then $TF_j' = LF_j - k$ .

Then, go back to step <3>.

2.3. The definitions of other time parameters are as same as those of CPM

$TF_{ij} = LF_j - ES_i - T_{ij}$

$SF_{ij} = LF_j - LF_i - T_{ij}$

$FF_{ij} = ES_j - ES_i - T_{ij}$
\[ IF_y = \max \left\{ 0, ES_y - LF_y - T_y \right\} \]

Same as that presented by Batterby (1967) and Thomas (1969).

\[ TF_y = LF_y - ES_y \]

Same as that presented by Elmaghraby (1977).

\[ ES_y = ES_i \]

\[ LF_y = LF_j \]

\[ EF_y = ES_i + T_y \]

\[ LS_y = LF_i - T_y \]

\[ TF_y = TF_i + SF_y = FF_y + TF_j \]
3. The correlated characteristic between the length and the float under GPRs

Denote $\mu_{uv}$ as one of the paths between node $(u)$ and $(v)$,

$$\mu_{uv} = (u) \rightarrow (a) \rightarrow (b) \rightarrow (c) \rightarrow \cdots \rightarrow (e) \rightarrow (f) \rightarrow (g) \rightarrow (v)$$

Its length, safety float and free float are as the follows:

$$\bar{\mu}_{uv} = T_{ua} + T_{ab} + T_{bc} + \cdots + T_{ef} + T_{fg} + T_{gv}$$

$$FF_{\mu_{uv}} = FF_{ua} + FF_{ab} + FF_{bc} + \cdots + FF_{ef} + FF_{fg} + FF_{gv}$$

$$SF_{\mu_{uv}} = SF_{ua} + SF_{ab} + SF_{bc} + \cdots + SF_{ef} + SF_{fg} + SF_{gv}$$

3.1. Theorem 1

$$ES_v - ES_u = \bar{\mu}_{uv} + FF_{\mu_{uv}} \quad (3)$$

Proof: According to the definition, $FF_{ij} = ES_j - ES_i - T_{ij}$, $\therefore$

$$FF_{\mu_{uv}} = FF_{ua} + FF_{ab} + FF_{bc} + \cdots + FF_{ef} + FF_{fg} + FF_{gv}$$

$$= (ES_u - ES_a - T_{ua}) + (ES_b - ES_a - T_{ab}) + \cdots$$

$$+ (ES_e - ES_f - T_{ef}) + (ES_g - ES_f - T_{fg}) + (ES_v - ES_g - T_{gv})$$

$$= ES_v - ES_u - T_{ua} - T_{ab} - T_{bc} - \cdots - T_{ef} - T_{fg} - T_{gv}$$

$$\therefore$$

$$FF_{\mu_{uv}} = ES_v - ES_u - \bar{\mu}_{uv}$$

$$\therefore$$

$$\bar{\mu}_{uv} + FF_{\mu_{uv}} = ES_v - ES_u$$

Therefore, equation (3) is true.

3.2. Theorem 2

$$LF_v - LF_u = \bar{\mu}_{uv} + SF_{\mu_{uv}} \quad (4)$$

According to the definition, $SF_{ij} = LF_j - LF_i - T_{ij}$, the proof is same as
Corollary 1: For any nodes \((u)\) and \((v)\)

\[
TF_v - TF_u = SF_{\mu_{uv}} - FF_{\mu_{uv}}
\]  

(5)

Proof: According to the definition:

\[
TF_v - TF_u = (LF_v - ES_v) - (LF_u - ES_u)
\]

\[
= (LF_v - LF_u) - (ES_v - ES_u)
\]

From equation (3) and (4):

\[
TF_v - TF_u = (\mu_{av} + SF_{\mu_{uv}}) - (\mu_{av} + FF_{\mu_{uv}})
\]

\[
= SF_{\mu_{uv}} - FF_{\mu_{uv}}
\]

Therefore, equation (5) is true.

Corollary 2: For any nodes \((u)\) and \((v)\), and any path \(\mu_{av}\),

If \(TF_v > TF_u\), then

\[
SF_{\mu_{av}} > FF_{\mu_{av}}
\]  

(6)

If \(TF_v < TF_u\), then

\[
SF_{\mu_{av}} < FF_{\mu_{av}}
\]  

(7)

Proof: (6) and (7) are true according to the equation (5).

Corollary 3: \(\mu_{uv}^\forall\) denote the longest path between node \((u)\) and \((v)\), \(\mu_{uv}^\forall\) denote its length; while \(\mu_{av}\) denote one of the paths between node \((u)\) and \((v)\), \(\mu_{av}\) denote its length. Then

\[
FF_{\mu_{av}} \leq FF_{\mu_{uv}}
\]  

(8)

\[
SF_{\mu_{av}} \leq SF_{\mu_{uv}}
\]  

(9)

Proof: According to equation (3), if the nodes \((u)\) and \((v)\)
confirmed, then \( ES_v - ES_u \) is confirmed, \( \therefore \)

\[
\bar{p}_{uv} + FF_{\mu_{uv}} = \bar{p}_{uv} + FF_{\mu_{uv}}.
\]

As \( \mu_{uv} \) is the longest path between node \((u)\) and \((v)\), \( \therefore \)

\[
\bar{p}_{uv} \geq \bar{p}_{uv}.
\]

\( \therefore \)

\[
FF_{\mu_{uv}} \leq FF_{\mu_{uv}}.
\]

Same as above,

\[
\bar{p}_{uv} + SF_{\mu_{uv}} = \bar{p}_{uv} + SF_{\mu_{uv}}.
\]

As \( \mu_{uv} \) is the longest path between node \((u)\) and \((v)\), \( \therefore \)

\[
\bar{p}_{uv} \geq \bar{p}_{uv}.
\]

\( \therefore \)

\[
SF_{\mu_{uv}} \leq SF_{\mu_{uv}}.
\]

Therefore, (8) and (9) are true.

**Corollary 4:** For any path \( \mu \) between the begin node and the end node, its safety float equals to its free float.

\[
FF_{\mu} = SF_{\mu}. \tag{10}
\]

Proof: In equation (5), Take the begin node as node \((u)\), then \( TF_u = 0 \), and take the end node as node \((v)\), then \( TF_v = 0 \), then equation (5) turned out as the follows:

\[
SF_{\mu} - FF_{\mu} = 0.
\]

\( \therefore \)

\[
FF_{\mu} = SF_{\mu}.
\]
Therefore equation (10) is true.

3.3. Theorem 3

If \((er) \in \mu_i^*\), then

\[ FF_{e_r} = 0 \]  

(11)

Proof: take \((u) = (1)\) into equation (3), then \(ES_i = 0\); take \((v) = (i)\), then \(\mu_i = \mu_i^*\), then

\[ ES_i = \overline{\mu}_i^* + FF_{\mu_i^*}. \]

From equation (1), \(ES_i = \overline{\mu}_i^*\), therefore \(FF_{\mu_i^*} = 0\).

As \((er) \in \mu_i^*\), therefore \(FF_{er} = 0\) according to the definition, equation (11) is true.

3.4. Theorem 4

If \((er) \in \mu_i^{\ominus}\), then

\[ SF_{e_r} = 0 \]  

(12)

Proof: using equation (4) and (2), and the method is same as 3.3.

3.5. Theorem 5

\[ FF_{\mu} = \sum_{(i) \neq \mu} FF_{i_j} = \overline{\mu}^\nu - \overline{\mu} \]  

(13)

Proof: Put \((u) = (1)\) into equation (3), then \(ES_u = 0\); Put \((v) = (w)\) into equation (3), \((w)\) is the end node, then \(ES_w = \overline{\mu}^\nu\), therefore \(\mu_w = \mu_{iw} = \mu\).

From equation (3), \(ES_w - ES_i = \overline{\mu}_{iw} + FF_{\mu_{iw}}\), \(\overline{\mu}^\nu = \overline{\mu} + FF_{\mu}\), that is
Therefore equation (13) is true.

3.6. Theorem 6

\[ SF_{\mu} = \sum_{(ij) \in \mu} SF_{ij} = \mu^v - \mu \]  

(14)

Proof: Equation (14) can be proved by equation (4) as above.

**Corollary 1:** If \( FF_{\mu} = \sum_{(ij) \in \mu} FF_{ij} = 0 \), then

\[ \mu^v = \mu \]  

(15)

Proof: Equation (15) can be proved by equation (13).

**Corollary 2:** If \( SF_{\mu} = \sum_{(ij) \in \mu} SF_{ij} = 0 \), then

\[ \mu^v = \mu \]  

(16)

Proof: Equation (16) can be proved by equation (14).

3.7. Theorem 7

\[ TF_{ij} = \mu^v - \mu_{ij}^v \]  

(17)

Proof: According to the definition, \( \mu_{ij}^v = \mu_{ij}^* + T_{ij} + \mu_{ij}^\theta \). From equation (1), \( \mu_{ij}^* = ES_i \),

From equation (2), \( \mu_{ij}^\theta = \mu^v - LF_j \), \( \therefore \)

\[ \mu_{ij}^v = ES_i + T_{ij} + (\mu^v - LF_j) \]

That is
\[ \bar{\mu}^\mu - \bar{\mu}^\nu = LF_j - ES_i - T_j \]

From definition, \( TF_j = LF_j - ES_i - T_j \), \( \therefore \)
\[ \bar{\mu}^\mu - \bar{\mu}^\nu = TF_j \]

Therefore equation (17) is true.

3.8. Theorem 8

\[ SF_{ij} = \bar{\mu}^\mu - \bar{\mu}^\nu \]  \hfill (18)

Proof: From definition, \( \bar{\mu}^\mu_i = \bar{\mu}^\mu_i + \bar{\mu}^\nu_j \), \( \bar{\mu}^\nu_j = \bar{\mu}^\nu_i + \bar{\mu}^\mu_j \), \( \therefore \)
\[ \bar{\mu}^\mu_i - \bar{\mu}^\nu_j = (\bar{\mu}^\mu_i + \bar{\mu}^\nu_j) - (\bar{\mu}^\mu_j + \bar{\mu}^\nu_j) \]
\[ = \bar{\mu}^\mu_i - \bar{\mu}^\nu_j - T_j \]
\[ = (\bar{\mu}^\mu_i - \bar{\mu}^\nu_j) - (\bar{\mu}^\nu_i - \bar{\mu}^\mu_j) - T_j \]

From equation (2), \( \bar{\mu}^\nu_i - \bar{\mu}^\nu_j = LF_j \), \( \bar{\mu}^\mu_i - \bar{\mu}^\mu_j = LF_i \), \( \therefore \)
\[ \bar{\mu}^\mu_i - \bar{\mu}^\nu_j = LF_j - LF_i - T_j \]
\( \therefore \)
\[ \bar{\mu}^\mu_i - \bar{\mu}^\nu_j = SF_{ij} \]

Then, equation (18) is true.

3.9. Theorem 9

\[ FF_{ij} = \bar{\mu}^\nu_i - \bar{\mu}^\nu_j \]  \hfill (19)

Proof: From definition, \( \bar{\mu}^\nu_j = \bar{\mu}^\nu_j + \bar{\mu}^\mu_i \), \( \bar{\mu}^\nu_i = \bar{\mu}^\nu_i + \bar{\mu}^\mu_j \), \( \therefore \)
\[ \bar{\mu}^\nu_j - \bar{\mu}^\nu_i = (\bar{\mu}^\nu_j + \bar{\mu}^\mu_i) - (\bar{\mu}^\nu_i + \bar{\mu}^\mu_j) \]
\[ = \bar{\mu}^\nu_j - \bar{\mu}^\nu_i - T_j \]

From equation (1), \( \bar{\mu}^\nu_j = ES_j \), \( \bar{\mu}^\nu_i = ES_i \), \( \therefore \)
\[ \bar{\mu}^\nu_j - \bar{\mu}^\nu_i = ES_j - ES_i - T_j \]
\[ \mu_j^\nu - \mu_j^\mu = FF_{ij} \]

Then, equation (19) is true.

3.10. Theorem 10

\[ TF_i = \mu_i^\nu - \mu_i^\mu \]  

(20)

Proof: From definition, \( \mu_i^\nu = \mu_i^\mu + \mu_i^\rho \), \( \therefore \)

\[ \mu_i^\nu - \mu_j^\nu = \mu_i^\nu - (\mu_i^\mu + \mu_i^\rho) = (\mu_i^\nu - \mu_i^\rho) - \mu_i^\mu \]

From equation (2) and (1), \( \mu_i^\nu - \mu_i^\rho = LF_i, \mu_i^\mu = ES_i \), \( \therefore \)

\[ \mu_i^\nu - \mu_i^\nu = LF_i - ES_i = TF_i \]

\[ \therefore \]

\[ TF_i = \mu_i^\nu - \mu_i^\nu \]

Then, equation (20) is true.

3.11 Theorem 11

Suppose \( \phi \) is a cycle, the cycle value (the length of the cycle)

\[ r = \sum_{\phi \in \varphi} T \]

then

\[ \sum_{\phi \in \varphi} FF_{ij} = -r \]

(21)

Proof: From (3) \( ES_v - ES_u = \mu_v^\mu - \mu_u^\beta = \mu_v^\mu + \sum_{(i,j) \in \omega} FF_{ij} \)

Let \( (u) = (v) \), then \( \mu_v^\mu = \bar{\phi} = r \), \( \sum_{(i,j) \in \omega} FF_{ij} = \sum_{i \neq \omega} FF_{ij} \)
\[ \therefore 0 = r + \sum_{\phi \in \varnothing} FF_{ij} \]
\[ \therefore \sum_{\phi \in \varnothing} FF_{ij} = -r \]

3.12 Theorem 12

\[ \sum_{\phi \in \varnothing} SF_{ij} = -r \]

.................................(22)

Proof: It can be proved by (4)

Corollary 1:

\[ \sum_{\phi \in \varnothing} FF_{ij} = \sum_{\phi \in \varnothing} SF_{ij} = -r \]

(23)

3.13 Theorem 13

If \( r = 0, (ij) \in \varnothing \), then

\[ EF_{ij} = SF_{ij} = 0 \]

(24)

\[ TF_i = TF_j = 0 \]

(25)

Proof:

From (23), when \( r = 0, (ij) \in \varnothing \), then \( FF_{ij} = SF_{ij} = 0 \), (24) is true.

\[ \therefore TF_{ij} = TF_{ij} + SF_{ij} = FF_{ij} + TF_j, \quad \therefore SF_{ij} = FF_{ij}, \quad \therefore TF_i = TF_j = 0 \]

(25) is true.

That is to say the node float is zero if it is on a zero cycle.

3.14 Theorem 14
If \( r = 0 \) \( , (u) \in \phi \) and \( TF_u = 0 \), Then, for any \( (ij) \in \phi \),

\[ TF_j = 0 \]  

(26)

Proof:

As \( (u) \in \phi , (ij) \in \phi \), when \( r = 0 \), from (25)

\[ TF_u = TF_i = TF_j = 0 \]

And \( \therefore r = 0 \), from (24), \( FF_y = SF_y = 0 \)

And \( \therefore \)

\[ TF_y = TF_i + SF_y = FF_y + TF_j \]

it can be proved that \( (TF_i + SF_y) = (FF_y + TF_j) = 0 \), \( \therefore \)

\[ TF_y = 0 \]

That is to say: if there is node with zero float on a cycle, then all the other nodes in this cycle will have zero float, all the activities in this cycle will have zero float.
4 The anomalies of main activity

4.0 Main activity and its partner activity

If the duration of activity B must be deducted by that of activity of A, and turns out to be $(T_B - T_A)$, then A is the main activity of activity B and B is its partner activity.

If B is the immediately preceding activity of A, then B is its preceding partner activity, noted as $^\circ B$;

If B is the immediately succeeding activity of A, then B is its immediately succeeding partner activity, noted as $C^\circ$;

If A has only preceding partner activity $^\circ B$, then this activity is called as forward-single main activity, and noted as $^\circ A$;

If A has only succeeding partner activity $C^\circ$, then this activity is called as Backward-single main activity, and noted as $A^\circ$;

If A has both $^\circ B$ and $C^\circ$, then this activity is called double main activity, denoted as $A$

Forward-single main activity $^\circ A$ and Backward-single main activity $A^\circ$ are all single activities and can be noted as $\bar{A}$.

4.1 The characteristic of normal main activity

4.1.1 Characteristic 1

The main activity is the only immediately succeeding activity of its preceding partner activity, and the only immediately preceding activity of its succeeding partner activity.

Proof: Suppose there is a preceding partner activity $H$ with two main immediately succeeding activities $H, A$, as shown in Fig.17. This supposition can be proved wrong as the follows.

As the duration of activity $H$ is $(T_H - T_A)$, activity $A$ is the main activity. Which means that “Activity $A$ will be finished no later than $T_H$ days after that of activity $C$”. The graphic representation is as Fig.18 without amendment.
The corresponding correct representation will be as Fig. 19 after amendment.

![Diagram](C→\(a_1\)→\(H\)\(→\(-T_H−T_A\)→\(A\)→\(a_2\)→\(D\))

Fig. 19

Therefore the partner activity \(H\) has no influence on activity \(R\). The \((-T_H−T_A)\) will vary with \(T_A\) and has nothing to do with \(B\). Fig. 17 is wrong.

It can be similarly proved that the following part of Characteristic 1 is also true.

4.1.2 Characteristic 2

If the main activity is not in a cycle, then its partner activity will not be in that cycle either.

Proof: As mentioned on 4.1.1, the main activity is the only immediately succeeding activity of its preceding partner activity and the only immediately preceding activity of its succeeding partner activity. Therefore, the Characteristic 2 can be proved easily.

4.2 The anomalies of the critical main activity

4.2.1 Concept

(1) **Forward-critical activity** \(A^+\) (Elmaghraby, 1992)

If \(T_A\) denotes the duration of activity \(A\), \(T\) denotes the project duration, and \(A^+\) the forward-critical activity.

If \(T'_A > T_A \Rightarrow T' > T\), and \(T'_A < T_A \Rightarrow T' < T\), then \(A = A^+\).

(2) **Backward-critical activity** \(A^-\) (Elmaghraby, 1992)

If \(T'_A > T_A \Rightarrow T' < T\), and \(T'_A < T_A \Rightarrow T' > T\), then \(A = A^-\).

(3) **Neutral critical activity** \(A^{(v)}\)

If \(T'_A > T_A \Rightarrow T' = T\), and \(T'_A < T_A \Rightarrow T' = T\), then \(A = A^{(v)}\).

(4) **F-N-B critical activity** \(A^{(+-)}\)

If (i) \(0 < T_A \leq e, A = A^+\), (ii) \(e < T_A \leq f, A = A^{(v)}\) and (iii) \(T_A > f, A = A^-\)

Then \(A = A^{(+-)}\). “e” is named as the forward threshold and f is its backward threshold.

(5) main activity and its partner activity

If the duration of activity \(H\) is \(-T_H−T_A\), then \(A\) is the main activity of \(H\), and \(H\) is its partner activity. The partner activity must be a relation activity.

If \(H\) is the immediately preceding partner activity of main activity \(A\), it is a preceding partner activity, denoted by \(^0H\).

If \(H\) is the immediately succeeding partner activity of main activity \(A\), it is a succeeding partner activity, denoted by \(^H\).

If \(A\) only has \(^0H\), we call it preceding single main activity, denoted by \(^{\Lambda}A\).
If A only has a $H^0$, we call it succeeding single main activity, denoted by $A^\wedge$.
If A has both $H^0$ and $^6H$, we call it double main activity, denoted by $A$.

$A^\Delta$ and $A^\hat{\Delta}$ are all single main activity, can be denoted as $A$.

4.2.2 The characteristic of Neutral critical activity $A^{(V)}$

4.2.2.1 The characteristic 1

The prolonging (shortening) of neutral critical activity $A^{(V)}$ has no influence on the project completion. So it will not be considered in the time-trade off problem.
Proof: it can be proved by the definition.

4.2.2.2 The characteristic 2

The neutral critical activity $A^{(V)}$ must has a preceding critical partner activity or a succeeding critical partner activity, but will never has both of them at the same time.

Proof: As the duration of the partner activity $B$ is $T_B - T_A$, and that of the main activity is $T_A$, the sum of them is a constant $T_B$, which has nothing to do with $T_A$. Therefore the project completion time will not vary with $T_A$. If the succeeding and preceding partner activities appear together, then the sum of $B, A, C$ will be $(T_B - T_A) + T_A + (T_C - T_A) = T_B - T_A + T_C$, which will vary with $T_A$ and have influence on the project completion time. This will lead to contradiction with the definition of neutral critical activity $A^{(V)}$.

4.2.2.3 The characteristic 3

The neutral critical activity $A^{(V)}$ may be single main activity $A$, or double main activity $\overline{A}$, that is $A^{(V)} = \overline{A}$ or $A^{(V)} = \overline{A}$.

(1) If $A^{(V)} = \overline{A}$, then the only partner activity will also be critical activity.

(2) If $A^{(V)} = \overline{A}$, then one of the partner activity is critical activity and the other one is non-critical.

Proof: (1) can be proved by definition and (2) by 4.2.2

4.2.3 The characteristic of Backward-critical activity $A^-$

4.2.3.1 The characteristic 1

If $T'_A > T_A$, then $T' < T$ and if $T'_A < T_A$ then $T' > T$.

Proof: The duration sum of activity $A^-$, preceding partner activity $B$ and the succeeding partner activity $C$ is as the follows:

$S = T_A + (T_B - T_A) + (T_C - T_A) = T_B - T_A + T_C$

When $T'_A > T_A$, that is $T'_A = T_A + \Delta t$, then $S' = S + \Delta t$,

When $T'_A < T_A$, that is $T'_A = T_A - \Delta t$, then $S' = S + \Delta t$.

4.2.3.2 The characteristic 2

The preceding partner activity $B$ and the succeeding partner activity $C$ of backward-critical activity $A^-$ must be both of critical activities.

Proof: If $B$ or $C$ is not a critical activity, then we can get $\overline{A} = A^{(V)} (\overline{A} \neq A^-)$, the project completion will not vary with $T_A$. 

4.2.3.3 The variation range of Back-critical activity $A^-$

(1) When $T_A' = T_A - \Delta t$, the project completion increases with it.

\[ 0 \leq \Delta t \leq \mu_A - L_A, \quad \mu_A - L_A \]
denotes the maximum value that activity A can be shortened.

(2) When $T_A' = T_A + \Delta t$, the project completion reduces with it.

\[ 0 \leq \Delta t \leq \min \{ FF_{i_t}, SF_{a_t} \} \]

\[ FF_{i_t} = \min \{ FF_{r_i} | (r_i) \not\in \mu^v \} \]

\[ SF_{a_t} = \min \{ SF_{a_t} | (a_t) \not\in \mu^v \} \]

$A^-$ = $(a_1,a_2)$

Proof: According to 4.1, when the main activity $A = (a_1a_2)$, then the immediately succeeding activity of $a_1$ is A, and $a_2$ has only A as its immediately preceding activity A, which is shown as Fig.20.

When $T_A' = T_A - \Delta t$, then

\[ T_{c'} = T_C - T' = T_C - (T_A - \Delta t) = T_C - T_A + \Delta t = T_C + \Delta t, \]

that is

\[ T_C' = T_C + \Delta t. \]

\[ C' \in \mu^v, \quad T' > T. \]

\[ TF_{a_t} > TF_{a_t}. \]

$a_t$ Has no influence on the project completion.

When $T_A$ reduce, $T_{a_b}$ will increase. \[ B \in \mu^v, \quad ES_{a_t} > ES_a, \quad TF'_{a} > TF_{a}, \]

Therefore $ra_1$ also has no influence on the project completion.

The project completion varies with the duration of $A^-$, which is the range of $I_A \leq T_A \leq U_A$, so

\[ \Delta t \leq U_A - I_A, \quad \text{and} \quad (1) \text{ is true.} \]

(2) When $T_A' = T_A + \Delta t$

\[ B = (a_1a_2) \not\in \mu^v, \quad (a_i) \not\in \mu^v, \quad \mu_{a_i} \subset \mu^v, \quad \text{and} \quad B = (a_1a_2) \in \mu^v. \]

When $T_A' = T_A + \Delta t$, then

\[ T_b = T_{b} - \Delta t. \]

\[ B \in \mu^v, \quad \mu_{a_i}^v = \mu_{a_i} - \Delta t. \]

When $\Delta t = FF_{r_{a_i}}$, then

\[ \mu_{a_i}^v = \mu_{a_i} - \Delta t = \mu_{a_i}^v - FF_{r_{a_i}}. \]
According to the definition, \( \vec{\mu}^* = ES_{a_1} \), \( FF_{ra_1} = ES_{a_1} - ES_{a_1} - T_{a_1} \)
\( \therefore \vec{\mu}^{a_1} = \vec{\mu}^* - FF_{ra_1} = ES_{a_1} - (ES_{a_1} - ES_{a_1} - T_{a_1}) = ES_{a_1} + T_{a_1} \)
\( \therefore A \in \mu^v \), \( \therefore a_2 \in \mu^v \), \( C^v \in \mu^v \), \( \therefore \mu^{a_2}_2 \subset \mu^v \) \( \mu^{a_2}_2 \in \mu^{a_2}_2 \) \( \therefore T_{C} = T_{A} + FF_{ra_1} \) \( T_{C} = T_{C} - T_{A} \)
\( \therefore T_{C} = T_{C} - FF_{ra_1} \), \( \vec{\mu}^{a_2}_2 = \vec{\mu}^{a_2}_2 - FF_{ra_1} \)
When \( FF_{ra_1} < SF_{a_2} \),
Then \( \vec{\mu}^{a_2}_2 = \vec{\mu}^{a_2}_2 - FF_{ra_1} > \vec{\mu}^{a_2}_2 - SF_{a_2} \)
According the definition \( LF_{a_2} = \vec{\mu}^* - \vec{\mu}^{a_2}_2 \), \( \therefore \mu^{a_2}_2 = \vec{\mu}^* - LF_{a_2} \)
And \( SF_{a_2} = LF_{a_2} - LF_{a_2} - T_{a_2} \), \( \therefore \mu^{a_2}_2 > \mu^{a_2}_2 - SF_{a_2} = (\vec{\mu}^* - LF_{a_2}) - (LF_{a_2} - LF_{a_2} - T_{a_2}) \)
\( \therefore \mu^{a_2}_2 > \vec{\mu}^* - LF_{a_2} + T_{a_2} \)
That is \( \vec{\mu}^{a_2}_2 > \vec{\mu}^{a_2}_2 + T_{a_2} \)
\( \therefore \) the new critical path \( \mu^{v} \) pass activity \( C^o \)
As \( A \in \mu^v \), therefore \( \mu^v \) pass activity \( A^- \) and \( C^o \), as it has been proved \( \mu^{a_1}_1 = ES_{C} + T_{a_1} \).

Then there is at least one new critical path \( \mu^{v} \) pass \( ra_1 \), \( A^- \) and \( C^o \), but not \( B \). In this path \( \mu^v \), both the main activity \( A^- \) and succeeding partner activity \( C^o \) exist. Therefore, when \( T_{A} \) continues increase, the length of \( \mu^v \) will not change, and the project completion will not change either. Therefore, when \( FF_{ra_1} < SF_{a_2} \), the maximum increase value of \( T_{A} \) should be \( \Delta t \leq FF_{ra_1} \).

When there are many activities same as \( ra_1 \), \( FF^{a_1}_1 = \min \left\{ FF_{ra_1} \mid (ra_1) \notin \mu^v \right\} \), then \( \Delta t \leq FF^{a_1}_1 \).

When \( FF_{ra_1} > SF_{a_2} \), it can be proved that the maximum increase value of \( T_{A} \) is
\( \Delta t \leq SF_{a_2} \).

When there are many activities same as \( a_2 \), \( SF^{a_2}_2 = \min \left\{ SF_{a_2} \mid (a_2) \notin \mu^v \right\} , 0 \leq \Delta t \leq \min \left\{ FF^{a_2}_2, SF^{a_2}_2 \right\} \).

4.2.3.4 The characteristic of Critical single main activity \( \vec{A} \)
(1) \( \vec{A} = \nabla A \), as Fig.21,
(i) The preceding partner activity \( \overleftarrow{B} \) of a certain critical forward single main activity must be a non-critical activity.

Proof: otherwise it will become a neutral critical activity \( \overleftarrow{A}^{(V)} \).

(ii) if the duration of \( \overleftarrow{A} \) reduced by \( \Delta t \), lead to the shortening of the project completion, the range of \( \Delta t \) is \( 0 \leq \Delta t \leq FF_{B} \).

Proof: as \( \overleftarrow{A} \in \mu^\overleftarrow{V} \), \( \overleftarrow{A} = (a_{1},a_{2}) \), so \( \mu_{a_{i}}^{*} = (1) \rightarrow \cdots \rightarrow (r) \rightarrow (a_{i}) \), so \( \overleftarrow{B} \not\in \mu_{a_{i}}^{*} \), \( \overleftarrow{B} = \mu_{a_{i}}^{\prime} \). According to the definition, \( ES_{a_{i}} = ES_{a_{i}} \). According to the definition, \( \mu_{a_{i}}^{*} = \mu_{a_{i}}^{\prime} \). When \( T_{A} = T_{A} - \Delta t \), \( T_{B} = T_{B} - T_{A} \), so \( T_{B}^{'} = T_{B} + \Delta t \) and \( \mu_{a_{i}}^{\prime} = \mu_{a_{i}}^{*} + T_{B}^{'} = (\mu_{a_{i}}^{*} + T_{B}) + \Delta t \). When \( T_{A} > FF_{B} = FF_{a_{a_{i}}a} \), then \( \mu_{a_{i}}^{*} > \mu_{a_{i}}^{\prime} + T_{B} + FF_{a_{a_{i}}a} \), according to the definition, \( \mu_{a_{i}}^{*} > ES_{a_{i}} + T_{B} + (ES_{a_{i}} - ES_{a_{i}} - T_{a,a_{i}}) \). As \( T_{a,a_{i}} = T_{B}^{'} \), so \( \mu_{a_{i}}^{*} > ES_{a_{i}} \). As \( \mu_{a_{i}}^{\prime} = \mu_{a_{i}}^{*} + T_{B}^{'} \), so the \( \overleftarrow{B} \) is on the new critical path. According to 4.1.2, the main activity \( A \) must be on the new critical path too, therefore the main critical activity \( A \) become a neutral critical activity \( A^{(V)} \), and the project completion will not be shorten if \( T_{A} \) reduced further.

When \( 0 \leq \Delta t \leq FF_{B} \), (ii) is true.

(iii) When \( T_{A}^{'} = T_{A} + \Delta t \), then \( T_{B} > T \).

Proof: when the duration of \( \overleftarrow{A} \) is increased, the duration of \( \overleftarrow{B} \) will reduce, which has no influence on the project completion. Therefore the characteristic of \( \overleftarrow{A} \) is as same as that of \( A^{*} \).

(2) \( \overleftarrow{A} = A^{V} \) shown as Fig22.

(i) The succeeding partner activity of \( A^{V} \) must be non-critical activity

Proof: otherwise \( A^{V} \) become a neutral critical activity \( A^{(V)} \).
(ii) $T_d$ increase and the project completion will prolong

Proof: if $T_d$ increased, $T_c' = T_c - T_d$ will reduce, which has no influence on the project completion. So the project completion has the same characteristic as $A^+$, will increase.

(iii) $T_d' = T_d - \Delta t$, $\Delta t > 0$, then $T' < T$, the range of $\Delta t$ will be $0 \leq \Delta t \leq SF_c$.

Proof: As $A^V \in \mu^V$, $A^V = (a, a_2)$, so $(a_2) \in \mu^V$, and $\mu^V_{a_2} = (a_2) \rightarrow (t) \rightarrow \cdots \rightarrow (w) \in \mu^V$.

According to the definition, $C^+ \notin \mu^V$, so $C^+ \notin \mu^V_{a_2}$, $C^+ = (a_2a_3)$. According the definition of the longest path, $\mu^{V^+}_{a_2} > \mu^{V^+}_{a_3} + T_c^-$. When $T_d' = T_d - \Delta t$, $\Delta t > 0$, as $T_c' = T_c - T_d$, $T_c' = T_c$, so

$$T_c' = T_c - T_d = T_c - (T_d - \Delta t)$$
$$= (T_c - T_d) + \Delta t$$
$$= T_c + \Delta t$$

so

$$T_c' = T_c + \Delta t$$

As $C^+ \notin \mu^V_{a_3}$, so $\mu^{V^+}_{a_3} = \mu^{V^+}_{a_3}$, and

$$\mu^{V^+}_{a_3} + T_c' = \mu^{V^+}_{a_3} + (T_c + \Delta t)$$

When $\Delta t > SF_c$, then

$$\mu^{V^+}_{a_3} + T_c' > \mu^{V^+}_{a_3} + T_c + SF_c$$

According to the definition, $LF_{a_3} = \mu^{V^-} - \mu^{V^+}_{a_3}$, so $\mu^{V^+}_{a_3} = \mu^{V^-} - LF_{a_3}$, as $SF_c = SF_{a_2a_3} = LF_{a_2} - LF_{a_3} - T_{a_2a_3} = LF_{a_2} - LF_{a_3} - T_c$, put it into the above equation, we get:

$$\mu^{V^+}_{a_3} + T_c' > (\mu^{V^-} - LF_{a_3}) + T_c + (LF_{a_2} - LF_{a_3} - T_c)$$
$$= \mu^{V^-} - LF_{a_2} = \mu^{V^+}_{a_3}$$

so

$$\mu^{V^+}_{a_3} + T_c' > \mu^{V^+}_{a_3}$$

Therefore, $C^+$ is on the new critical path $\mu^{V^+}$. According to 4.1.2, the main activity A will be also on the $\mu^{V^+}$, then A become a neutral critical activity $A^{(V)}$, when $\Delta t$ increased further, $T' = T$. Therefore, $\Delta t \leq SF_c$ for the lowest cost, so

$$0 \leq \Delta t \leq SF_c$$

(iii) is true.

4.2.4 The characteristic of F-N-B critical activity $A^{(\pm)}$

The characteristic of F-N-B critical activity $A^{(\pm)}$ is shown as Fig.23.
4.2.4.1 Characteristic 1

$A^{(+-)}$ is double main activity, and its preceding partner activity $^+B$ and succeeding partner activity $^+C$ are both non-critical activity.

Proof: if both $^+B$ and $^+C$ are critical activities, then $A^{(+-)}$ become a $A^-$. If one of $^+B$ or $^+C$ is critical activity, then A is neutral critical $A^\oplus$. Therefore, $^+B$ and $^+C$ are both non-critical activity.

4.2.4.2 Characteristic 2

If the duration of F-N-B critical activity $A^{(+-)}$ is increased, then the project completion will be prolonged.

Proof: When the duration of $A^{(+-)}$ is increased, the duration of $^+B$ and $^+C$ reduce. As they are both non-critical activities, their change has nothing to do with the project completion. So $A^{(+-)}$ has the normal characteristic like $A^-$. Therefore, $T_t$ will increase if $T_A$ increases.

4.2.4.3 Characteristic 3

Its characteristic when the duration of $A^{(+-)}$ is reduced.

Let $T'_A = T_A - \Delta t$, $\Delta t > 0$, $A^{(+-)} = (a_1a_2)$, $^+B = (a_0a_1)$, $^+C = (a_2a_3)$,

<1> When $0 \leq \Delta t \leq \min \{FF_B, SF_C\}$, then $(T - T') \leq \Delta t$.

<2> When $\Delta t > \min \{FF_B, SF_C\}$, then $(T - T') \leq \min \{FF_B, SF_C\}$.

Proof: As Fig.23, the critical path is $\mu^V = (1)\rightarrow\cdots\rightarrow (r)\rightarrow (a_1)\rightarrow (a_2)\rightarrow (t)\rightarrow\cdots\rightarrow (w)$;

The longest path passing $^+B$ is $\mu^V_B = (1)\rightarrow\cdots\rightarrow (a_0)\rightarrow (a_1)\rightarrow (a_2)\rightarrow (t)\rightarrow\cdots\rightarrow (w)$;

The longest path passing $^+C$ is $\mu^V_C = (1)\rightarrow\cdots\rightarrow (r)\rightarrow (a_1)\rightarrow (a_2)\rightarrow (a_3)\rightarrow\cdots\rightarrow (w)$.

$1> \overline{p^V}_B = \overline{\mu^V} - FF_B$, $\mu^V_C = \overline{\mu^V} - SF_C$.

Because $TF_B = TF_{a_0a_1} = FF_{a_0a_1} + TF_{a_1}$, $A = (a_1a_2) \in \mu$, $\therefore (a_1) \in \mu^V$, $\therefore TF_{a_1} = 0$.

$\therefore TF_B = FF_{a_0a_1} = FF_B$.

According the theory of float as the follows,

$TF_B = \overline{\mu^V} - \overline{p^V}_B$.

From above equation, $FF_B = \overline{\mu^V} - \overline{p^V}_B$, $\therefore \overline{p^V}_B = \overline{\mu^V} - FF_B$.

Similarly, if can be proved that $\mu^V_C = \overline{\mu^V} - SF_C$. 
When $A' = -\Delta$, $B = T_A - \Delta$, $C = T_A + \Delta$, then $BB = \mu\nabla$, $CC = \mu\nabla$.

It is known: $BA = T_A - \Delta$, $CA = T_A + \Delta$.

When $A' = -\Delta$, then $B = T_A - \Delta$, $C = T_A + \Delta$, $T_A + T_B = (T_A - \Delta) + (T_A + \Delta) = (T_A + T_B)$

Similarly, $T_A + T'_B = (T_A + T_B)$.

$\therefore B \in \mu_B, A \in \mu_A, \forall T_A + T'_B = (T_A + T_B)$, $\therefore \mu_B' = \mu_B$.

And $\therefore C' \in \mu_C, A \in \mu_A, T_A + T'_C = (T_A + T_C)$, $\therefore \mu_C' = \mu_C$.

$3> \text{let } FF = FF_B = FF_{a_B}, SF = SF_C = SF_{a_C}, \text{ and let } FF^* < SF^*$, then $FF_B < SF_C$.

From the conclusion in 1$: \mu_B = \mu - FF_B, \mu_C = \mu - SF_C, \therefore FF_B < SF_C$.

When $0 \leq \Delta t \leq \min \{FF^*, SF^*\}$, then $0 \leq \Delta t \leq FF_B$.

From the conclusion in 1$\mu_B = \mu - FF_B \leq \mu - \Delta t$.

It can be proved that $\mu_B' > \mu_C', \therefore \mu' \geq \mu - \Delta t, \therefore \mu - \mu' \leq \Delta t, \therefore \mu = T$.

$\therefore T - T' \leq \Delta t$, $T - T'$ varies with $\Delta t$.

<2> When $\Delta t > \min \{FF^*, SF^*\}$, that is $\Delta t > FF_B$.

From the conclusion in 1$\mu_B = \mu - FF_B > (\mu - \Delta t)$. It has been proved that $\mu_C > \mu_B, \therefore \mu_C' > \mu_B$, $\therefore \mu - \mu' < \mu - \mu_B = FF_B$.

$\therefore \mu = T$, $\therefore T - T' < FF_B = \min \{FF^*, SF^*\}$

That is to say $(T - T')$ will be less than a constant $FF_B$, no matter how much $\Delta t$ is.

When $FF^* > SF^*$, similarly, <1>, <2> can be proved as true.

Example, Fig24.
In Fig. 24, the duration of critical activity A is shortened from 16 to 14 days, the project completion is shortened by 2 days also, as Fig. 25.

If we continually shorten the durations of activity A by another 2 days, from 14 to 12, But the project completion is not changed this time.

If we continually shorten the duration of activity A, from 12 to 6 days, then the project
completion is prolonged by 4 days, from 58 to 62 days, as shown in Fig. 27.

Case study:

According to Fig. 24,

\[
FF^* = FF_j = (ES_j - ES_i - T_j) = 30 - 34 - (-6) = 2
\]

\[
SF^* = SF_{\text{av}} = LF_v - LF_u - T_{\text{av}} = 40 - 46 - (-12) = 6
\]

\[
\min \{ FF^*, SF^* \} = \min \{ 2, 6 \} = 2
\]

\[
\max \{ FF^*, SF^* \} = \max \{ 2, 6 \} = 6
\]

When the duration of A is reduced from 16 to 14 days, \( \Delta t = 2 \leq \min \{ FF^*, SF^* \} \).

According to 4.2.4.3-<1>, the project completion will reduce from 60 to 58 days.

When the duration of A is further reduced from 14 to 12 days,

When \( \min \{ FF^*, SF^* \} < \Delta t = 4 < 6 = \max \{ FF^*, SF^* \} \)

According to 4.2.4.3-<2>, the project completion is not changed, remain as 58 days.

When the duration of A is further reduced from 12 to 6 days,

\( \Delta t = 10 > 6 = \max \{ FF^*, SF^* \} \).

According to 4.2.4.3-<3>, the project completion will increase from 58 to 62 days.

4.3 The Characteristic of non-critical double main activity \( \overline{A} \).

Only the duration shortening of the non-critical double activity \( \overline{A} \), can led to the project completion prolonging, which can be used in resource leveling problem. We will discuss this next.

4.3.1 The characteristic of non-critical activity \( \overline{A} \).

<1> when the duration of non-critical double activity is increased, it has the same characteristic as other common non-critical activity, and the project completion will not increased.

<2> if the duration of the non-critical double activity \( \overline{A} \) is reduced by \( \Delta t \) days, then

<3> When \( 0 < \Delta t \leq (FF_{\overline{A}} + TF_{\overline{A}} + SF_{\overline{A}}) \), the project completion is not influenced, that is \( \Delta T = T' = T \).
<ii> When $\Delta t > \left( FF_B + TF_A + SF_C \right)$, the project completion is prolonged by $\left[ \Delta t - \left( FF_B + TF_A + SF_C \right) \right]$, that is $T' = T - \Delta t - \left( FF_B + TF_A + SF_C \right)$.

Proof: As Fig 28

![Diagram](image)

Fig:28

<1> when the $\Delta t$ increases, both $T_B = (T_B - T_A)$ and $T_C = (T_C - T_A)$ will reduce, and will not affect the project completion. When $T_A$ increase, $\bar{A}$ has the same characteristic as the common activity, that is, when $\Delta t \leq TF_A$, $T' = T$.

When $\Delta t > TF_A$, then $T' > T$.

<2> when $T_A$ is reduced, both $T_B = (T_B - T_A)$ and $T_C = (T_C - T_A)$ will increase, then their sum $S = T_A + T_B + T_C = (T_B + T_C - T_A)$ will also increase, which may affect the project completion.

According to the definition:

$FF_{a_0a_1} + TF_{a_1a_2} + SF_{a_2a_3} = (ES_{a_0} - ES_{a_1} - T_{a_0a_1}) + (LF_{a_1} - ES_{a_1} - T_{a_0a_1}) + (LF_{a_2} - LF_a - T_{a_2a_3})$

$= (\mu^{\nabla} - \mu^{\nabla}_a) - \mu^{\nabla}_{a_0} - (T_{a_0a_1} + T_{a_1a_2} + T_{a_2a_3})$

$= \mu^{\nabla} - (\mu^{\nabla}_{a_0} + T_{a_0a_1} + T_{a_1a_2} + T_{a_2a_3} + \mu^{\nabla}_a)$

Suppose the longest path passing $'B$, $\bar{A}$ and $C'$ is $\mu^{\nabla}_{BAC}$.

According to the definition:

$\mu^{\nabla}_{BAC'} = \mu^{\nabla}_{a_0} + T_{a_0a_1} + T_{a_1a_2} + T_{a_2a_3} + \mu^{\nabla}_{a_3}$

$= \mu^{\nabla}_{a_0} + (T_{a_0a_1} + T_{a_1a_2} + T_{a_2a_3}) + \mu^{\nabla}_{a_3}$

$\therefore FF_B + TF_A + SF_C = \mu^{\nabla} - \mu^{\nabla}_{BAC'}$

$\therefore$ non of $'B$, $\bar{A}$, $C'$ in on the $\mu^{\nabla}_{a_0}$ and $\mu^{\nabla}_{a_3}$, $\therefore \mu^{*}_{a_0} = \mu^{*}_{a_3}$, $\mu^{*}_{a} = \mu^{*}_{a}$.

As we already get as above: $S = T_A + T_B + T_C = (T_B + T_C - T_A)$, $T_B = T_B$, $T_C = T_C$, $T' = T_B$, $T_C' = T_C$.

$S' = (T_B' + T_C' - T_A') = T_B + T_C - (T_A - \Delta t)$

$= (T_B + T_C - T_A) + \Delta t = S + \Delta t$
That is \( S' = S + \Delta t \). \( \therefore T'_B + T'_{A} + T'_{C} = (T_B + T_A + T_C) + \Delta t \).

\( \therefore \) when \( T' = T - \Delta t \),

\[
\mu'_{BAC} = \mu_{BAC} + \Delta t
\]

\[
\mu_{BAC} = \mu_{BAC} + \Delta t
\]

\( \therefore \) when \( \Delta t \leq FF_B + TF_A + SF_C \), then

\[
\mu_{BAC} \leq \mu_{BAC} + \Delta t
\]

It can be getting from above that:

\[
\mu_{BAC} = \mu_{BAC} + \Delta t
\]

When the \( T_A \) reduced, other path passing \( \mu_B < \mu_{BAC} \), so the project completion is not affected.

\( \therefore \) when \( \Delta t > FF_B + TF_A + SF_C \),

It can be get from above that, \( \mu_{BAC} > \mu_{BAC} \), so the project completion will be prolonged, and

\[
T' = T + \Delta t
\]

\[
T = \mu_{BAC} + \Delta t
\]

\[
\mu_{BAC} = \mu_{BAC} + \Delta t
\]

According to the above conclusion,

\[
FF_B + TF_A + SF_C = \mu_{BAC} + \Delta t
\]

From \( <i> \) and \( <ii> \), \( <2> \) is true.

**Example:**

In Fig.29, \( A = (ju) \), \( TF_{ju} = LF_{ju} - ES_{j} - T_{ju} = 45 - 30 - 15 = 3 \), \( B = (ij) \), \( FF_{ij} = ES_{j} - ES_{i} - T_{ij} = 30 - 34 - (-5) = 1 \), \( C = (uv) \).
\( SF_{uv} = LF_v - LF_u - T_{uv} = 42 - 48 - (-8) = 2 \),
\[ \therefore (FF_B + TF_A + SF_{C^*}) = 1 + 3 + 2 = 6 \]

When \( T_A' = T_A - \Delta t \),

\(<i>\) when \( \Delta t = 2 \), \( T_A \) is reduced from 15 to 13 days, \( \therefore \Delta t = 2 < 6 = FF_B + TF_A + SF_{C^*} \).

According to \(<i>\), the project completion will not change and will remain 62 days as shown in Fig.30.

\(<ii>\) when \( \Delta t = 10 \), \( T_A \) is reduced from 15 to 5 days, \( \therefore \Delta t = 10 > 6 = FF_B + TF_A + SF_{C^*} \), according to \(<ii>\) \( T' - T = 10 - 6 = 4 \), the project completion will be \((62 + 4) = 66\) days, as shown in Fig.31.
5. Theory of non-positive cycle optimization

In the problem of shortening the project completion at the lowest cost under GPR, the network with cycle can be treated as that with no cycle, and no positive cycle will appear during the whole process of optimization.

Proof: In order to shorten the project completion, only the duration of those critical activities should be considered, shortening or prolonging. As long as the lowest cost is guaranteed, there will no positive cycle appearing in any situation, with cycle or not.

(1) when a certain critical activity A is not a main activity.

Only the duration of activity A be reduced, can the project completion be shortened. As $T_A > 0$, if $T_A$ reduced, the cycle value passing A will not increase, so there will be no positive cycle appear.

(2) when the critical activity is a main activity

1) when its partner activity is also a critical activity, then A is a neutral critical activity $A^{(v)}$. According to 4.2.2, the duration of A has no influence on the project completion. Therefore, the duration of A will not be changed during optimization, so no positive cycle will appear.

2) when its partner activity is not a critical activity

If the partner activity of A is not on the cycle of A, then there will be no positive cycle appearing. If the partner activity of A is on the cycle of A, they must be on the same cycle due to the conclusion mentioned on 4.1.2. The length of the cycle will not change according to the conclusion mentioned on 4.2.2.1, therefore, there will also be no positive cycle.

From 1) and 2), critical single main activity will no led to positive cycle during optimization.

(3) when the critical activity A is a double main activity

1) when the A’s partner activities $B$ and $C$ are both of non-critical activity, then A is F-N-B critical activity $A^{(v-1)}$. $T_A = T_A - \Delta t$, there will be no positive cycle. According to 4.2.4.2, $T_A$ increase, $T$ will not reduce, so $T_A$ should be shortened. According to 4.2.4.3, the maximum value that $T_A$ can be shortened is $0 \leq \Delta t \leq \min \{FF*, SF*\}$.

(i) If both $B$ and $C$ are not in cycle, then, there will be no positive cycle with the shortening of $T_A$.

(ii) If only one of $B$ and $C$ is in the cycle, then A must be in the same cycle according to 4.1.2. According to 4.2.3.4 (1)-(ii) and (2)-(iii), the shortening of $T_A$ will not increase the cycle value, so there will be no positive cycle.

(iii) If both $B$ and $C$ are in cycles,

When $0 \leq \Delta t \leq \min \{FF*, SF*\}$, there will be no positive cycle.

1) If $0 \leq \Delta t \leq \min \{FF*, SF*\} = FF^* = FF_{B} = FF_{a_1a_2}$, Let $SF^* = SF^*_{c} = SF^*_{a_2a_3}$, $A = (a_1, a_2)$, As shown in Fig.32.
Suppose the cycle pass “B, A and C” is as the follows:
\[(a_0) \rightarrow (a_1) \rightarrow (a_2) \rightarrow (a_3) \rightarrow (a_4) \rightarrow \ldots \rightarrow (a_n) \rightarrow (a_0)\]
According to the definition, \(FF_{ij} = ES_j - ES_i - T_j\), \(FF_{ij} = FF_{a_i a_j}\), then
\[
FF_{0,1} + FF_{1,2} + FF_{2,3} + FF_{3,4} + \cdots + FF_{n,0} \\
= (ES_1 - ES_0 - T_1) + (ES_2 - ES_1 - T_2) + (ES_3 - ES_2 - T_3) + \cdots + (ES_n - ES_{n-1} - T_n)
\]
So
\[
\Delta t \leq \min \{FF^*, SF^*\} = FF_{a_0 a_1} = FF_{0,1} \\
< FF_{0,1} + FF_{1,2} + FF_{2,3} + FF_{3,4} + \cdots + FF_{n,0} \\
= -(T_{0,1} + T_{1,2} + T_{2,3} + T_{3,4} + \cdots + T_{n,0})
\]
That is
\[
\Delta t < -(T_{0,1} + T_{1,2} + T_{2,3} + T_{3,4} + \cdots + T_{n,0}) = \text{value of the cycle}
\]
so
\[
\Delta t + (T_{0,1} + T_{1,2} + T_{2,3} + T_{3,4} + \cdots + T_{n,0}) < 0
\]
When \(T_A' = T_A - \Delta t\), \(T_A' + T_B' + T_C' = T_A + T_B + T_C + \Delta t\), therefore, when \(T_A\) reduced by \(\Delta t\), the value of cycle will increase by \(\Delta t\), but it is still a negative cycle.

2. If \(0 \leq \Delta t \leq \min \{FF^*, SF^*\} = SF^* = SF_{a_i a_j}\), According to the definition, \(SF_{ij} = LF_j - LF_i - T_j\), it can be proved that
\[
SF_{0,1} + SF_{1,2} + SF_{2,3} + SF_{3,4} + \cdots + SF_{n,0} = -\left( T_{0,1} + T_{1,2} + T_{2,3} + T_{3,4} + \cdots + T_{n,0} \right)
\]
so
\[
\Delta t < -\left( T_{0,1} + T_{1,2} + T_{2,3} + T_{3,4} + \cdots + T_{n,0} \right)
\]
That is
\[
\Delta t + \left( T_{0,1} + T_{1,2} + T_{2,3} + T_{3,4} + \cdots + T_{n,0} \right) < 0
\]
If \( T_A \) reduced by \( \Delta t \), the value of cycle will increase by \( \Delta t \), but it will still be negative cycle.

From (i), (ii), (iii), (1) is true.

2) When only one of \( A \)'s partner activities (\( B \) and \( C^+ \)) is critical activity, then \( A \) is a neutral activity \( A^{(V)} \), the project completion will not vary with \( T_A \). Therefore \( A \) will not be used during the whole process of optimization, and there will be no positive cycle.

3) When both of \( A \)'s partner activities (\( B \) and \( C^+ \)) are critical activity, then, \( A \) is backward-critical activity, \( A = A^- \). The shortening of project completion can only be acquired by prolonging the duration of \( A \), that is \( T'_A = T_A + \Delta t, \Delta t > 0 \).

(i) If \( ^B, ^C \) and \( A \) are all in a same cycle, then, as \( T'_A = T_A + \Delta t \), it can be proved easily that \( T'_A + T'_B + T'_C = \left( T_A + T_B + T_C \right) - \Delta t \), there will be no positive cycle.

(ii) If only one of them (\( ^B \) and \( ^C^+ \)) is in the same cycle of \( A \), then \( A \) turns out to be like the forward-critical activity in cycle, the value of the cycle will not change and there will be no positive cycle.

(iii) If neither \( ^B \) nor \( ^C^+ \) is in the same cycle of \( A \), and \( A \) is in a cycle.

1) As there is no \( A \)'s partner activity in the cycle, according to 4.2.3.3 (2), If \( T_A \) increased by \( \Delta t \), when the project completion is shortened, the maximum range of \( \Delta t \) is as the follows:

\[
0 \leq \Delta t \leq \min \left\{ FF^*, SF^* \right\}
\]

\[
FF^* = \min \left\{ FF_{r:a_i} \mid (ra_i) \not\in \mu^V \right\}
\]

\[
SF^* = \min \left\{ SF_{a:tt} \mid (a:tt) \not\in \mu^V \right\}
\]

Using the method in 5- (3) - (1) - (ii), It can be proved that: if \( T_A \) increased by \( \Delta t \), the value of the cycle will increase by \( \Delta t \), which is less than the value of the positive cycle, and there will be no positive cycle.

2) If \( ^B \) and \( ^C^+ \) are in the cycle, as \( T'_A = T_A + \Delta t \), then \( T'_A + T'_B + T'_C = \left( T_A + T_B + T_C \right) - \Delta t \), so the value of the cycle will reduce and there will be no positive cycle.

3) If only one partner activity is in the same cycle of \( A \), the sum of the main activity and the partner activity will remain as a constant, the value of the cycle will not change and no positive cycle will appear.
(4) From the conclusion of 4.1.2, it is impossible if "B or C" are not in a same cycle of A.

From (1), (2), (3) and (4), the theory is true.
6 Application

6.1 Network simplification based on the Float Theory

6.1.1 Principle

Suppose the project completion should be shortened by $\Delta T$ days, from $T$ to $(T - \Delta T)$.

If $TF_{ij} \geq \Delta T$, Activity $(ij)$ can be deleted from the original network $G$, and a sub-network $D$ is acquired. We can then reduce the project completion by $\Delta T$ days in the sub-network, which have a same result as that we get in the original network $G$.

Proof: According the equation (17) of the Float Theory:

$$TF_{ij} = \bar{\mu} - \bar{\nu}, \cdots$$

$$\bar{\mu}_{ij} = \mu - TF_{ij}$$

As it is known that $TF_{ij} \geq \Delta T$, $\bar{\nu}_{ij} \leq (\mu - \Delta T)$.

As $T = \bar{\mu}$, $\bar{\mu}_{ij} \leq (T - \Delta T)$, It will have no influence on the project completion $(T - \Delta T)$ if we delete $\bar{\mu}_{ij}$ from the original network. According to $\bar{\mu}_{ij} \leq \bar{\mu}_{ij}$, all the path $\mu_{ij}$ passing activity $(ij)$ will be deleted if we delete the activity itself. As $\bar{\mu}_{ij} \leq \bar{\mu}_{ij} \leq (T - \Delta T)$, this deletion has no influence to target the project completion $(T - \Delta T)$, $D$ and $G$ will have the same result.

6.1.2 Method

<1> In the original network $G$, the time parameter $ES_i$ and $LF_i$ should be work out according to 2.1, 2.2.

<2> Calculate the $TF_i$ of each node $(i)$, and delete those nodes and activities under the condition $TF_i \geq \Delta T$.

<3> Calculate the total float of the balance activities by $TF_{ij} = LF_j - ES_i - T_{ij}$, delete all the activities under the condition $TF_{ij} \geq \Delta T$, we get $D$.

Proof: According the theory of non-positive cycle optimiaztion and the Ford method, <1> is true.

\[ \therefore \text{when } TF_i \geq \Delta T, \text{ then } \]

\[ TF_{ki} = FF_{ki} + TF_i \geq TF_i \geq \Delta T. \]

Similarly,

\[ TF_{ij} = TF_i + SF_{ij} \geq TF_i \geq \Delta T \]

According to 6.1.1, it can be proved that the deletion of $(ki)$, $(ij)$ and $(i)$ has no influence on the project completion, so <2> is true.

According to 6.1.1, <3>is true.
6.2 The longest path algorithm (Fulkerson) (1964) under GPRs

6.2.1 Feasibility analysis

Using AoA graphic representation, and the graphic method in this article to represent the “lower and upper bounds” relation, the main activity and its partner activities will be in order, there will be no double lines between two nodes, like \( i \xrightarrow{k} j \). This is helpful for us to develop the longest path algorithm under GPRs.

As the realtion activity in not true activity, which can not be shortened. We set their cost rate as \( a_y = \infty \), then it can be treated as normal activity during the optimization.

6.2.2 The longest path algorithm in amendment network (Fulkerson) (1964) under GPRs

<1> looking for the critical path \( \mu^V \) in network \( D, \ D_k = G \).

<2> treat the cost rates \( a_y \) of each activity as capacity, and let the \( \mu^V \) have the largest flow.

<3> amend the capacity of each activity according to Table 1

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{The flow situation} )</td>
</tr>
<tr>
<td>( &lt;i&gt; \ f_y = 0 )</td>
</tr>
<tr>
<td>( &lt;ii&gt; \ 0 &lt; f_y &lt; a_y )</td>
</tr>
<tr>
<td>( &lt;iii&gt; \ f_y = a_y )</td>
</tr>
<tr>
<td>( &lt;iv&gt; \ f_y &gt; a_y )</td>
</tr>
</tbody>
</table>

(a) Forward arrow, the direction of the arrow on the path is as same as that of the original network.

(b) Backward arrow, the direction of the arrow on the path is opposite to that of the original network.

\[ a_y \] denote the lowest cost rate of activity \((ij)\), \( f_y \) denote the flow passing \((ij)\)

The above steps are as same as those in CPM.

<4> According to tabel 2, amend the duration of the activiy on the critical path, we get a amendment network \( D_{k+1} \).

<5> Using \( D_{k+1} \) instead of \( D_k \), fo back to <1>.

<table>
<thead>
<tr>
<th>Table 2 (CPM)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{Flow} )</td>
</tr>
<tr>
<td>( &lt;i&gt; \ f_y = 0 )</td>
</tr>
</tbody>
</table>
\[ u_j \text{ denote the maximum duration (with the lowest cost), } l_j \text{ denote the minimum duration (the highest cost), } a_j \text{ capacity, } f_j \text{ flow.} \]

**Table 2 (GPRS)**

<table>
<thead>
<tr>
<th>Flow</th>
<th>( T_y ) reduce by ( \Delta t &gt; 0 )</th>
<th>( T_y ) increase by ( \Delta t &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>&lt;i&gt; 0 ( f_j ) = 0</td>
<td>( i \rightarrow j )</td>
<td>( i \rightarrow j )</td>
</tr>
<tr>
<td>&lt;ii&gt; 0 ( f_j &lt; a_j )</td>
<td>( i \rightarrow j )</td>
<td>( i \rightarrow j )</td>
</tr>
<tr>
<td>&lt;iii&gt; ( f_j = a_j )</td>
<td>( i \rightarrow j )</td>
<td>( i \rightarrow j )</td>
</tr>
<tr>
<td>&lt;iv&gt; ( f_j &gt; a_j )</td>
<td>( i \rightarrow j )</td>
<td>( i \rightarrow j )</td>
</tr>
</tbody>
</table>

\( T_y \) denote the duration of \((ij)\) under the lowest cost, \( T_y = u_j \), \( f_j \) flow, \( a_j \) denote the positive cost rate when the duration reduced, and the absolute value of negative cost rate when the duration increase.

\( \Delta t \) can be get according to Table 3.

\[ FF_{a_1}^* = \min \left\{ FF_{a_1} \mid (ra_1) \notin \mu^v \right\} \]

\[ SF_{a_2}^* = \min \left\{ SF_{a_2} \mid (a_2,t) \notin \mu^v \right\} \]

**Table 3**

<table>
<thead>
<tr>
<th>Critical activity ( A = (a_1,a_2) )</th>
<th>Partner activity ( B, C^0 )</th>
<th>The maximum value to be shortened</th>
<th>( T_A' )</th>
<th>Maximum ( \Delta t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A^- ) ( 0B, C^0 )</td>
<td>( 0B ) critical &amp; ( C^0 ) critical</td>
<td>( T_A + \Delta t )</td>
<td>( u_A - l_A )</td>
<td>( \min \left{ FF_{(a_1)}^<em>, SF_{(a_2)}^</em> \right} )</td>
</tr>
<tr>
<td>( A^{(v)} ) either ( 0B ) critical or ( C^0 ) critical</td>
<td></td>
<td>( T_A )</td>
<td></td>
<td>( (u_A - l_A) )</td>
</tr>
<tr>
<td>( A^+ ) Non</td>
<td>( 0B ) non-critical</td>
<td>( T_A - \Delta t )</td>
<td>( u_A - l_A )</td>
<td>( \min \left{ FF_{(a_1)}^*, (u_A - l_A) \right} )</td>
</tr>
<tr>
<td>( v \cdot A ) ( 0B ) non-critical</td>
<td>( 0B ) non-critical</td>
<td>( T_A )</td>
<td>( u_A - l_A )</td>
<td>( \min \left{ SF_{(a_2)}^*, (u_A - l_A) \right} )</td>
</tr>
<tr>
<td>( A^v ) ( C^0 ) non-critical</td>
<td>( C^0 ) non-critical</td>
<td>( T_A )</td>
<td>( u_A - l_A )</td>
<td>( \min \left{ SF_{(a_1)}^*, (u_A - l_A) \right} )</td>
</tr>
</tbody>
</table>
\[ A^{(\rightarrow)} \quad 0^B \text{ non-critical} \quad 0^C \text{ non-critical} \quad T_A - \Delta t \quad \min \{ FF_{gB}, SF_{c0}, (u_A - l_A) \} \]

Proof: <1> <2> <3> is true as it is according to traditional method.

The <4> step, duration amendment. The key is to get the maximum value that a activity can be shortened. In CPM, the maximum value is \( \Delta t = (u_A - l_A) \), the maximum duration \( u_g \) in GPRS, in backward-critical activity, as the duration is prolonged, that is \( T'_A = T_A + \Delta t \), so the maximum duration is \( u_g = T'_g = T_g + \Delta t \), \( l_g = T_g \), during the duration amendment, For example, in <iii> \( f_{ij} = a_{ij} \). It should be \( i \to j \), so it is amended as \( i \to j \).

For \( A^* \), \( A^V \), \( A^\rightarrow \) and \( A^{(\rightarrow)} \), the duration should be shortened, that is \( T'_A = T_A - \Delta t \), therefore the maximum duration is \( u_g = T'_g = T_g - \Delta t \), and the confirmation of the maximum value to be shortened.

\[ \begin{align*}
\text{situation <iv>, } & f_{ij} > a_{ij}, \text{ it should be } i \to j, \text{ it is amended as } i \to j.
\end{align*} \]

The difference between GPRS and CPM is the activity with anomalies, such as \( A^* \), \( A^V \), \( A^\rightarrow \), and the confirmation of the maximum value to be shortened.

\[ \begin{align*}
\text{from 4.2.3.4<1>, } & \text{ } A^V \Delta t \leq FF_{gB}, \text{ as } \Delta t \leq (u_A - l_A) \\
\therefore & \Delta t = \min \{ FF_{gB}, (u_A - l_A) \}.
\end{align*} \]

\[ \begin{align*}
\text{from 4.2.3.4<2>, } & \text{ } A^{(\rightarrow)} \Delta t \leq SF_{c0}, \text{ as } \Delta t \leq (u_A - l_A) \\
\therefore & \Delta t = \min \{ SF_{c0}, (u_A - l_A) \}.
\end{align*} \]

\[ \begin{align*}
\text{from 4.2.4.3, } \Delta t \leq \min \{ FF_{gB}, SF_{c0} \}, \text{ as } \Delta t \leq (u_A - l_A) \\
\therefore & \Delta t = \min \{ FF_{gB}, SF_{c0}, (u_A - l_A) \}.
\end{align*} \]

\[ \begin{align*}
\text{from 4.2.3.3<2>, } \Delta t = \{ FF_{(u_A)} \}, SF_{(u_A)} \}
\end{align*} \]

\[ \begin{align*}
\text{for } A^*, \Delta t = (u_A - l_A).
\end{align*} \]

Therefore <4> is true.

In order to realize the maximum value to be shortened, iteration is necessary, so <5> is correct. Therefore the method is true.
6.3 Secondary critical path evaluation ($\mu^{[i]}$) and algorithm

6.3.1 Importance

When the difference \( \left( \overline{\mu^v} - \overline{\mu}^{[i]} \right) \) between the critical path \( \mu^v \) and the secondary critical path is small, \( \mu^{[i]} \) is easy to become a \( \mu^v \) due to the bad use of float. Therefore, the \( \mu^{[i]} \) should be paid great attention. If \( \left( \overline{\mu^v} - \overline{\mu}^{[i]} \right) \) is big, then \( \mu^{[i]} \) will not be that important.

It is very important to evaluate \( \left( \overline{\mu^v} - \overline{\mu}^{[i]} \right) \). A simple method is present in this article to calculate \( \left( \overline{\mu^v} - \overline{\mu}^{[i]} \right) \) with the theory of float. Also, a method is give to looking for the \( \mu^{[i]} \).

6.3.2 The evaluation theory

The critical path is \( \mu^v \), the secondary path is \( \mu^{[i]} \), let \( A_i = (a,u) \), \( (u) \in \mu^v \), \( (a) \notin \mu^v \),

\[
FF_A = \min \left\{ FF_{A_i} \mid A_i = (a,u), FF_{A_i} > 0, u \in \mu^v \right\}, \text{ then } \overline{\mu^v} - \overline{\mu}^{[i]} = FF_A
\]

Proof:

\[
\begin{align*}
\text{Fig 33} \\
1 & \xrightarrow{a_1} \cdots \xrightarrow{a_i} U \xrightarrow{a_j} \cdots W
\end{align*}
\]

All the paths in G pass a certain \( A_i \), \( FF_A > 0 \), \( \{ \mu_A \} \) is the set covering all the path passing \( A_i \), then \( G = \{ \mu_A \} + \{ \mu_A \} + \cdots + \{ \mu_A \} + \mu^v \). Suppose \( \overline{\mu^v} = \max \{ \mu_A \mid FF_{A_i} > 0 \} \).

According to equation, \( TF_{a,u} = TF_A = \overline{\mu^v} - \overline{\mu}^{[i]} \).\( \star \)

As \( TF_{a,u} = FF_{a,u} + TF_u \), it is known that \( (u) \in \mu^v \cdot TF_u = 0 \), \( \therefore TF_{a,u} = FF_{a,u} = FF_A \), \( \therefore \)

\[
FF_A = \overline{\mu^v} - \overline{\mu}^{[i]}
\]

That is

\[
\overline{\mu^v} = \overline{\mu}^{[i]} - FF_A
\]

It is known \( FF_A = \min \left\{ FF_{A_i} \mid A_i = (a,u), FF_{A_i} > 0, (u) \in \mu^v \right\}, \therefore \)

\[
FF_A \leq FF_A, \quad i = 1, 2, \cdots, n
\]

From above equation,

\[
\overline{\mu^v} \geq \overline{\mu}^{[i]}, \quad i = 1, 2, \cdots, n
\]

\( \therefore FF_A > 0 \), \( \therefore \overline{\mu^v} \neq \overline{\mu}^{[i]}, \overline{\mu^v} \neq \overline{\mu}^v \), according to the definition, \( \overline{\mu^v} = \overline{\mu}^{[i]} \), \( \therefore \)
\[ \overline{\mu} - \overline{[1]} = \overline{\mu} - \mu_{A} = FF_{A_1}. \]

6.3.3 Evaluation method

<1> calculate the time parameters of each node in G.

<2> Using \( FF_{a,u} = ES_{a} - ES_{u} - T_{a,u} \) to calculate \( FF_{a,u} \). \( (u) \in \mu^{V}, \ (a_{i}) \notin \mu^{V} \).

<3> Calculate \( FF_{A_1} = \min \{ FF_{a,u} \mid (a_{i}) \notin \mu^{V}, (u) \in \mu^{V}, FF_{a,u} > 0 \} \).

<4> Then \( \overline{\mu} - \overline{[1]} = FF_{A_1} \).

6.3.4 The method to looking for the secondary critical path

Suppose \( FF_{A_1} = \min \{ FF_{A_1} \mid A_1 = (a,u), (u) \in \mu^{V}, (a_{i}) \notin \mu^{V}, FF_{A_1} > 0 \} \), \( A_1 = (a,u) \).

<1> Looking for \( \mu_{a_{i}}^{*} \).

<i> looking for the node \( (e) \):
\( ES_{e} + T_{ea_{i}} = ES_{a_{i}} \)

<i> Looking for the node \( (f) \):
\( ES_{f} + T_{fe} = ES_{e} \)

......

<i>i> Till \( ES_{(i)} + T_{1,i} = ES_{i} \), stop

<iv> \( \mu_{a_{s}}^{*} = (1) \rightarrow (s) \rightarrow \cdots \rightarrow (f) \rightarrow (e) \rightarrow (a_{r}) \).

<2> Looking for \( \mu_{u}^{\oplus} = \mu_{u-w} = \mu^{V} \)

<3> the secondary critical path

\( \mu^{[1]} = \mu_{a_{i}}^{*} + (a_{i},u) + \mu_{u}^{\oplus} = (1) \rightarrow (s) \rightarrow \cdots \rightarrow (f) \rightarrow (e) \rightarrow (a_{r}) \rightarrow (u) \rightarrow \cdots (w) \)

Proof : \( \therefore ES_{e} + T_{ea_{i}} = ES_{a_{i}} \therefore ES_{a_{i}} - ES_{e} - T_{ea_{i}} = 0 \).

From definition: \( FF_{ea_{i}} = ES_{a_{i}} - ES_{e} - T_{ea_{i}} = 0 \).

Similarly, \( FF_{f_{e}} = 0, \cdots, FF_{1,s} = 0, \therefore \)

\( FF_{\mu_{u}}^{*} = FF_{1s} + \cdots + FF_{f_{e}} + FF_{ea_{i}} = 0 \)

From definition \( \mu_{u}^{\oplus} = \mu_{u}^{V-w} \) as \( (u) \in \mu^{V}, \therefore \mu_{u}^{\oplus} \in \mu^{V}, \therefore (ij) \in \mu_{u}^{\oplus}, \text{then } (ij) \in \mu^{V}, \)

\( \therefore TF_{ij} = FF_{ij} = 0, \therefore \)

\( FF_{\mu_{u}}^{\oplus} = \sum_{(ij) \in \mu_{u}^{\oplus}} FF_{ij} = 0 \)

\( \therefore \mu_{A_{1}}^{V} + (a_{i},u) + \mu_{u}^{\oplus}, \therefore FF_{\mu_{A_{1}}}^{V} = FF_{\mu_{u}}^{\oplus} + FF_{a_{i},u} + FF_{\mu_{u}}^{\oplus} \).

It is known from above, \( FF_{\mu_{u}}^{V} = FF_{a_{i},u} = FF_{A_1} \), according to equation (13),

\( FF_{\mu_{u}}^{V} = \overline{\mu} - \overline{\mu_{A_{1}}} \), \( \therefore \)

\( \overline{\mu} - \overline{\mu_{A_{1}}} = FF_{A_1} \).
From 6.3.2, \( \overline{\mu} - \overline{[\mu]} = F | F_{A_h} \), \( \therefore \overline{[\mu]} = \overline{\mu} \), that is \( \mu^{[1]} = \mu^{\overline{\mu}} \),

Therefore,

\[
\mu^{[1]} = \mu_{A_h}^{\overline{\mu}} = \mu_{A_h}^* + (a, u) + \mu_u^\oplus
\]

Step (1), (2) and (3) are true.
7 Case study

As Fig.33. It is required that the project completion to be shortened by 20 days at the lowest cost.
(1) Calculate the time parameters of each node $ES_i$ and $LF_i$, and $TF_i = LF_i - ES_i$.

(2) Delete all the nodes which is under condition $TF_i \geq \Delta T = 20$.

(3) Calculate the time parameters of each activity $TF_j = LF_j - ES_j - T_j$, delete those activities under condition $T_j > 20$. We get Fig 34. In order to guarantee the maximum value of project shortening to be 20 days, we add an additional activity of $(1,19)$, let $T_{1,19} = 180$.

Fig 34
(4) Critical path

\[ \mu^\gamma = (1) \rightarrow (5) \rightarrow (8) \rightarrow (10) \rightarrow (11) \rightarrow (14) \rightarrow (15) \rightarrow (16) \rightarrow (19), \]

let it have the biggest flow \( f_1 = 2 \), the flow graphic is as Fig.35, capacity to be amended according to table 1.

(5) Duration amendment according to Table 2.
As \((11,15)\) have preceding partner activity \((4,11)\), therefore \(FF_B = FF_{4,11}\).

\[
FF_{4,11} = ES_{11} - ES_4 - T_{4,11} = 70 - 100 - (-40) = 10
\]

So the \(\Delta t\) of \(T_A\) is

\[
\Delta t = \min\{FF_{4,11}, u_{4,11} - t_{4,11}\} = \min\{10, 100 - 70\} = 10
\]

So \(T_y = u_y\), \(T_y - \Delta t = 100 - 10 = 90\).
For $11,15$, as $f_{11,15} = 2 = a_{11,15}$, it can be get from table 2 that $\mu'=\mu$. Let it have the bigest flow, $f_2'=1 = a_{1,5} = 1$, capacity amendment according to table 1, we get Fig.37, then $f = f_1 + f_2 = 2 + 1 = 3$.

(6) The critical path $\mu''$ is as same as that of $\mu$. Let it have the bigest flow, $f_2' = 1 = a_{1,5} = 1$, capacity amendment according to table 1, we get Fig.37, then $f = f_1 + f_2 = 2 + 1 = 3$.

Fig 37

According to the flow in Fig 37, duration amendment according to table 2. As the flow of $(11, 15)$ is $f_{11,15} = 3 > a_{11,15} = 2$, according to table 2, it is amended as $\Delta t = 60 - 40 = 20$, $T_{1,8} - \Delta t = 40$, it can be got
that, $T_{4,11} = 60 - T_4 = 60 - 90 = -30$.

Fig 38

(8) New critical path $\mu^{''} = (1) \rightarrow (4) \rightarrow (11) \rightarrow (15) \rightarrow (16) \rightarrow (19)$, $f'' = a_{15,16} = 2$, its biggest flow is $f'' = 2$. Capacity amendment according to table 1, we get Fig39..
(9) From Fig 39, duration amendment according to table 2, we get Fig 40

As \( f'' = 2 = a_{15,16} \), so \((15,16)\) is amended as \(\downarrow\). The duration of \((4,11)\) is \(T_{4,11} = -30\), the forward direction is \(-30\), the backward direction is \(-(-30) = 30\), that is \((4,11)\) is amended as \(\downarrow\).
(10) we got two new critical path

$$\mu_1^\nu = (1) \to (4) \to (11) \to (16) \to (19)$$

$$\mu_2^\nu = (1) \to (3) \to (9) \to (17) \to (19)$$

Let them have the biggest flow and the capacitv is amended according to table 1, we got Fig 41.
（11）From Fig 41, duration amendment according to tabel 2, we get Fig.42

As $f_1 = a_{16,19} = 1$, so $(16,19)$ is amended as $16 \rightarrow 19$; 

As $f_{15,16} = 6 > a_{15,16} = 5$, so $(15,16)$ is amended as $15 \rightarrow 16$; 

As $f_2 = a_{3,9} = 5$, so $(3,9)$ is amended as $3 \rightarrow 9$;
(12) we get new critical path $\mu^{**} = (1) \rightarrow (19)$.

(13) Calculate the best value to be shortened for each activity when the project completion is 180 days.

Activity $(16, 19)$, $TF_{16,19} = LF_{19} - ES_{16} - T_{16,19} = 180 - 165 - 10 = 5$. Critical activity $(16, 19)$ should also be critical activity, now it has float time $TF_{16,19} = 5$, it means that
Activity (16,19) has been shortened (from 20 to 10 days) more than it is necessary. Therefore (16,19) can be prolonged by 5 days again, then it becomes a new critical path. So the best value to be shortened for (16,19) is 20 - (10 + 5) = 5 days.

Activity (11,14), \( TF_{11,14} = TF_{11,15} = LF_{15} - ES_{11} - T_{11,15} = 160 - 70 - 90 = 0 \), it is still a critical path, which means that it is best value to be shortened. So the best value to be shortened for (11,14) is 100 - 90 = 10 days.

Activity (3,9), \( TF_{3,9} = LF_{9} - ES_{3} - T_{3,9} = 80 - 40 - 30 = 10 \), it can be a critical activity after it is prolonged by 10 days. Therefore, the best value to be shortened for (3,9) is 45 - (30 + 10) = 5 days.

Activity (1,5), \( TF_{1,5} = LF_{5} - ES_{1} - T_{1,5} = 60 - 0 - 40 = 20 \), (1,5) can be a critical activity after it is prolonged by 20 days. Therefore, the best value to be shortened for (1,5) is 60 - (40 + 20) = 0.

Activity (15,16), \( TF_{15,16} = LF_{16} - ES_{15} - T_{15,16} = 165 - 160 - 5 = 0 \), Its best value to be shortened is 10 - 5 = 5 days.

The result is as the table below.

<table>
<thead>
<tr>
<th>Activity to be shortened</th>
<th>(1,5)</th>
<th>(3,9)</th>
<th>(11,14)</th>
<th>(15,16)</th>
<th>(16,19)</th>
</tr>
</thead>
<tbody>
<tr>
<td>best value</td>
<td>0</td>
<td>5</td>
<td>10</td>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>
8 Conclusion

(1) We designed the graphic method of GPRs, which is as same as that of CPM network. This graphic representation is not only simple, but also it can reflect the regular pattern how the relation activity varied with the main activity.

(2) The correlation principle between the float and the path is summarized under GPRs.

(3) Confirmed the range that the anomaly activity have influence on the project completion.

(4) The Theory of non-positive cycle optimization is proved.

During the whole process of looking for the lowest shortening schedule, no positive cycle will appear under GPRs.

(5) A method to simplify the network is acquired with the above theory.

(6) Introduce the longest path algorithm in amendment network to the GPRs network

(7) A simple method is get to evaluate the secondary critical path.